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M.

AREAS AND VOLUMES

ELEMENTARY PLANE TRIGONOMETRY

BY

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CONTENTS

The Tangent of an Acute Angle. The Sine and Cosine of an Acute Angle. Miscellaneous Application of the Sine, Cosine and Tangent. The Co-tangent, Secant, and Coscant of an Acute Angle. Exercises on the Six Trigonometrical Ratios. Miscellaneous Revision Examples. Solution of Triangles. Miscellaneous Problems. Areas of Triangles, etc. Circular Measure. The Sphere. Trigonometrical Ratios of Angles greater than 180° . Directed Lines. Compound Angles. Graphs of Periodic Functions. Appendix I. Traverse Table. Appendix II. : Haversines. Miscellaneous Examples. Answers to Examples. *Index*.

SOME PRESS OPINIONS

Journal of Education.—"An excellent book for boys who have in view a career in which a practical knowledge of trigonometry is required. Much pains are taken to lay the foundation securely. Numerical calculation is put in the foreground, and numerous examples are given of the proper way of using books of tables, while repeated injunctions are given regarding the necessity of orderly arrangement and of the application of checks in order to secure accuracy. The problems discussed are chiefly such as occur in surveying, navigation, engineering, and physics."

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AREAS AND VOLUMES

THEIR ACCURATE AND
APPROXIMATE DETERMINATION

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PREFACE

As Mathematics is generally taught nowadays, a boy or girl learns at an early stage many facts which he or she accepts as intuitive or else as laid down by authority (this intuition or authority being fortified, probably, by experimental or numerical verification), but without being very clear as to the chain of reasoning which may connect these facts. Then comes a later stage when these facts should be gathered up into groups, and the logical connection between the members of each group worked out.

The facts of the mensuration of areas and solids are learnt, some from arithmetic books or lessons, some from geometry, others from trigonometry. The object of this little book is to bring together these facts, to hang them on a logical chain, and to deduce some consequences. A knowledge of methods of approximating to the areas under curves is demanded in some examinations, and is necessary to the student of Applied Mechanics and Engineering. Some teachers are finding that pupils, particularly those who have not previously studied the Calculus, are often very hazy about these approximate methods, while those who reach the integration stage in their Calculus studies are often not very clear as to the underlying concept of an area contained by a curve.

In Chapters I. and II. it is insisted that the determination of areas and volumes comes eventually always to the counting of "elementary" squares or cubes. Chapter III. develops this idea, and leads on to the usual approximate methods for estimating areas. These methods are applied in the last two chapters.

The book is, thus, in no sense a first book on Mensuration, but it should serve as a revision of this subject, in so far as areas and volumes are concerned. Chapters III. to V. contain all that the ordinary student will require to know in the way of approximation to areas and volumes, and of the application of such methods to mechanical problems. The chapters can be worked by a student with no knowledge of Calculus methods, and they will serve to inculcate the ideas which are fundamental to integration.

The authors are aware that all the explanations given in Chapters I. and II. are not above criticism. Simplicity of style has been aimed at, and it is almost impossible to write quite accurately on such a subject as the area bounded by a curved line without some "shorthand," such as the notation of Limits. They hope that readers will gather the right notions, even though these have been incompletely expressed.

The authors wish to express their indebtedness to their colleagues at Dartmouth, and to Mr. M. Megson of Rugby School, for criticisms and suggestions.

D. F. F.

H. E. P.

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AREAS AND VOLUMES

CHAPTER I

SOME SIMPLE AREAS

§ 1. **The Measurement of Areas.**—Areas are measured in *square units*, a square inch, square centimetre, etc., being respectively the area of a square of side one inch, one centimetre, etc.

From this definition it follows directly that the area of a rectangle, whose sides are respectively a and b units, where a and b are both whole numbers, is ab square units. For we can divide the area, in the manner of a chess-board, into a rows of unit squares, there being b such squares in each row. Fig. 1 shows this for the case in which a is 2 and b , 3.

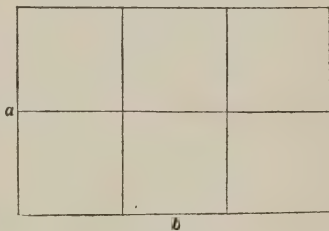


FIG. 1.

If we take a unit square—say, in which each side measures 1 inch—and divide each side into 10 equal parts, we can make a chess-board of 100 small squares. The length of the side of each small square is $\frac{1}{10}$ inch. Similarly, 100^2 , or 10^4 small squares, the side of each

of which is $\frac{1}{100}$ inch, can be exactly packed into one square inch.

Suppose, then, we have a rectangle of which the sides measure 3.4 inches and 4.56 inches respectively. If we divide each side into hundredths of an inch, there will be 340 of these in one side and 456 in the adjacent side. Completing the chess-board as before, we have 340×456 small squares. But we have shown that 100^2 of such small squares go to the square inch. Thus the number of square inches in the area of the rectangle is $\frac{340 \times 456}{100^2}$, or $\frac{340}{100} \times \frac{456}{100}$, or 3.4×4.56 square inches.

Arguing in this way, we see that the area of a rectangle, whose adjacent sides are a inches and b inches, is always ab square inches, whether a and b are whole numbers or whether they are fractions or decimals equivalent to fractions.

[*Note on the Application to Rectangles, the Sides of which are Incommensurable.*—The wording of the last sentence of the paragraph above may have appeared to be unnecessarily guarded. It seems to imply that there are rectangles to which the rule does not apply. The difficulty is this:

Suppose we have a rectangle of which the sides are $\sqrt{2}$ inches and $\sqrt{3}$ inches. Now, $\sqrt{2}=1.414 \dots$ and $\sqrt{3}=1.732 \dots$ neither of the decimals terminating or recurring. By the above argument we know that, if we take the sides as 1.41 inches and 1.73 inches, the area is 1.41×1.73 square inches; if we take the sides as

1.414 inches and 1.732 inches, the area is 1.414×1.732 square inches, and so on, to any degree of accuracy. So we conclude that the area of the rectangle considered is "exactly" $\sqrt{2} \times \sqrt{3}$ square inches.

This argument is convincing to "the ordinary person." The exact mathematician, however, would demand a more complete proof than this. There is always this difficulty of dealing precisely with incommensurable numbers like $\sqrt{2}$ and π . The point is, not that there is any doubt that a rectangle of which the sides are a inches and b inches contains exactly ab square inches of area, whatever a and b may be, but that it is not easy to give a short proof which will satisfy the rigid mathematician, in the case when a and b are incommensurable. All we have done is to make it "extremely probable."]

§ 2. From the area-rule for a rectangle the areas of certain other plane figures can be derived:

(i.) **Triangle.**—By geometry we know that the area of a triangle is one-half that of a rectangle, on the same base and between the same parallels. Thus, if b be the number of units of length in the base, and h the

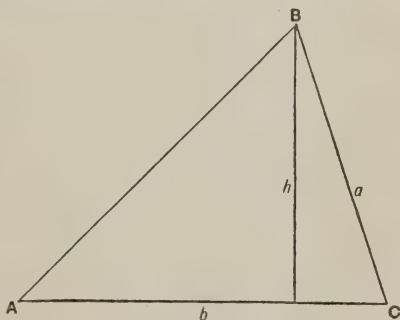


FIG. 2.

number of units in the vertical height (*i.e.*, the length

of the perpendicular from the vertex on to the base), the area is $\frac{1}{2}bh$ square units.

This can be expressed in the form $\frac{1}{2}ab \sin C$ (since, from Fig. 2, $h=a \sin C$).

In words, this rule may be expressed: "One-half the product of two sides, and the sine of the included angle."

Another rule for the area of a triangle, deducible from the above, is $\sqrt{s(s-a)(s-b)(s-c)}$, where s means $\frac{1}{2}(a+b+c)$.

The proof of this can be found in any book on trigonometry.

(ii.) **Trapezium.**—PQRS is a trapezium, the lengths of the parallel sides PQ and RS being a and b respectively; h is the length of the perpendicular distance between them. Join PR.

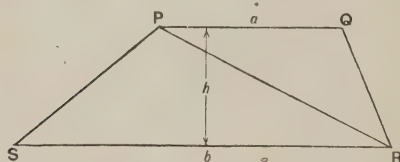


FIG. 3.

The area of the triangle PQR is $\frac{1}{2}ah$, and the area of the triangle PRS is $\frac{1}{2}bh$. But the sum of these triangles is the trapezium; hence, the area of the trapezium is $\frac{1}{2}h(a+b)$ square units.

In words: "One-half the sum of the parallel sides multiplied by the perpendicular distance between them."

(iii.) **Regular Polygon.**—The area is best found by joining each angular point to the centre of the figure.

This cuts the figure up into the same number of isosceles triangles as there are sides to the polygon. The area of each of these can be found, and thus the area of the whole figure.

Examples : (a) Regular hexagon of side, 2.3 cm.

If the centre is joined to each angular point, since $360^\circ \div 6 = 60^\circ$, it follows that the hexagon can be divided into 6 equilateral triangles. The area of each of these is $\frac{1}{2} \times 2.3 \times 2.3 \times \sin 60^\circ$ sq. cm., and the total area is six times this.

This comes to $3 \times 2.3^2 \times \sin 60^\circ$ —i.e., 13.7(5) sq. cm.

(b) Regular pentagon of side, 1.6 inches.

We have now five triangles, each of them having a vertical angle of $360^\circ \div 5 = 72^\circ$. The vertical height of each triangle (Fig. 4) is, therefore, $0.8 \times \cot 36^\circ$. The area of each triangle is, therefore, $0.8^2 \times \cot 36^\circ$.

The area of the pentagon is thus $5 \times 0.8^2 \times \cot 36^\circ$

$$\begin{aligned} &= 3.2 \times \cot 36^\circ \\ &= 4.40(4) \text{ sq. in.} \end{aligned}$$

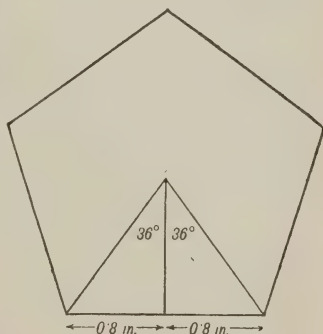


FIG. 4.

§ 3. The Area of a Sector of a Circle.—This can be

deduced from the rule for the area of a triangle as follows:

O is the centre of the circle and BA the arc bounding the sector. Take two points P and Q along the

arc, roughly trisecting it. The tangents at A, P, Q, B, are drawn, thus forming ACDEB, a portion of a polygon circumscribed to the circle.

Area of triangle AOC = $\frac{1}{2}$ OA $\times$ AC = $\frac{1}{2}$ $r \times$ AC (where r stands for length of radius of circle).

Area of triangle COD = $\frac{1}{2}$ OP $\times$ CD = $\frac{1}{2}$ $r \times$ CD.

Area of triangle DOE = $\frac{1}{2}$ OQ $\times$ DE = $\frac{1}{2}$ $r \times$ DE.

Area of triangle EOB = $\frac{1}{2}$ OB $\times$ BE = $\frac{1}{2}$ $r \times$ BE.

Thus, by addition, area of figure OACDEBO (which is a sector of a polygon circumscribed to the circle)

$$= \frac{1}{2} r \times (AC + CD + DE + EB).$$

= $\frac{1}{2}$ r multiplied by (that portion of the perimeter of the polygon lying between A and B).

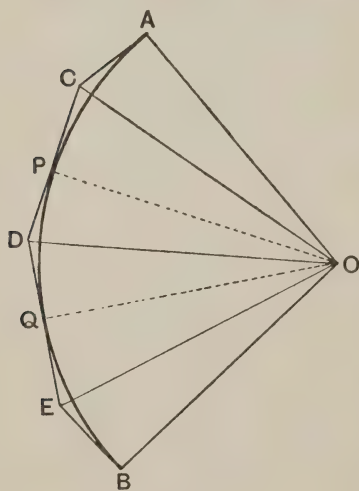


FIG. 5.

The above argument does not depend on the number of points we selected on the arc between A and B at which to draw tangents. It would have been equally true if we had taken 20 or 200 such points instead of 2. But the more points we take, the more closely does the portion of the perimeter of the polygon ap-

proximate to the length of the arc AB and, at the same time, the more closely does the area of the sector of the polygon approach the area of the sector of the circle.

So we deduce that the number of square units in the area of the sector of the circle OAB is given by half the product of the radius of the circle and the arc bounding the sector, these being measured in the same units of length.

As a particular case, if the bounding arc becomes the whole circumference, we have:

$$\begin{aligned}\text{Area of complete circle} &= \frac{1}{2}r \times \text{circumference} \\ &= \frac{1}{2}r \times 2\pi r \\ &= \pi r^2\end{aligned}$$

[*Note.*—We are here brought up against a difficulty. What do we mean by saying that the area of a circle is exactly πr^2 square inches? The difficulty is, of course, that π is one of the “incommensurable” numbers referred to in the Note to §1. Its value cannot, therefore, be expressed exactly by any terminating decimal. Yet the area of the circle is a perfectly definite thing. An answer to the difficulty is that a number, though incommensurable, is yet perfectly definite. But it will help towards a clear understanding of the later part of this book if we investigate this point a little more fully.

Suppose we draw a circle of radius 1 inch on squared paper grained in tenths of inches, and count up the squares contained within the boundary of the figure. Counting them carefully, we make out that there are 276 such squares (each of them of area $\frac{1}{100}$ th of a square inch) wholly contained within the area of the circle, and that the circumference of the circle cuts through 64 such squares. Some of these squares it almost

entirely encloses; of others it cuts off a very small portion.

All we can say definitely about the area enclosed by the circle, estimated by this method, is that it is greater than 276 such squares—*i.e.*, than 2.76 square inches—and less than $276 + 64$ such squares—*i.e.*, than 3.40 square inches.

If we assume that on the average each square that is cut through counts as half a square, we shall get the result 3.08 square inches. All we can say, however, with any certainty, is that the area of the circle is in the neighbourhood of 3 square inches.

Suppose we now take more finely-grained squared paper, in which each inch is divided into fiftieths. Counting squares as before, we get that the area of the circle lies between 7,708 and 8,000 such squares. Since 2,500 such squares go to the square inch, this means that the area lies between 3.083 and 3.200 square inches.

If we proceeded in this way, using squared paper more and more finely grained, we should gradually narrow down the limits between which the area of the circle is known to lie—that is to say, we should determine in order, with certainty, the figures after the decimal point of the number which represents the area of the circle expressed in square inches. The result πr^2 for the area of a circle means that these figures, as we obtain them, will be those of the number 3.14159. . . .

In practice, of course, we cannot carry this process of square-counting very far, as we cannot obtain nor

use very finely grained squared paper. We can dodge the difficulty to some extent by using a circle of radius greater than 1 inch. The figures above were obtained by using paper ruled in tenths of an inch, and a circle of radius 5 inches. The reader is advised to do this for himself. It is a process that throws light on the meaning of the area enclosed by curved lines, which will be considered at length in Chapter III.]

§ 4. **Right Prisms, Pyramids, Cylinders, Cones.**—The surface areas of prisms and pyramids need not be specially considered, for they are made up of rectangles, triangles, and polygons.

If we have a right cylinder of radius r and height h , and we imagine the surface to be slit along one of the

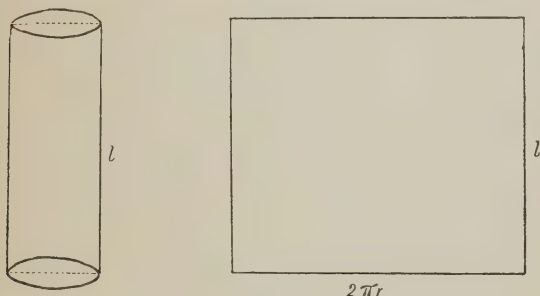


FIG. 6.

lines parallel to the axis, and then unrolled, it becomes a rectangle. One of the sides of this rectangle is h , and the other is the circumference of the cross-section—*i.e.*, $2\pi r$.

Thus the area of the curved surface is $2\pi rh$ square units.

If we add the areas of the top and bottom, we

get for the total surface-area of a closed cylinder— $2\pi r^2 + 2\pi rh$, or $2\pi r(r+h)$.

If we have a right cone, the radius of the base of which is r , and the slant (*i.e.*, the length of the line joining the vertex to any point of the base) l , and if

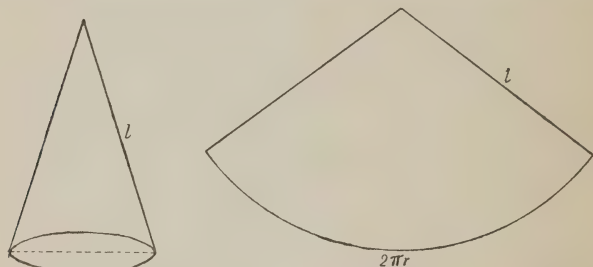


FIG. 7.

the cone be slit along one of the slants and the curved surface unrolled, it becomes a sector of a circle (since all the slants are equal).

The radius of this sector is l , and the arc is the same curved line that was originally the circumference of the base. Its length is, therefore, $2\pi r$.

Using the rule for the area of a sector given in § 3—*i.e.*, one-half the product of the length of the radius and the length of the arc—we have:

$$\text{Surface area} = \frac{1}{2}l \times 2\pi r = \pi rl.$$

For the total surface of a solid cone we should add πr^2 , the area of the base.

§ 5. Surface of a Sphere.—The problem of determining the surface of a sphere is not one that can easily be tackled by elementary methods, though it

is easy by Calculus. The root difficulty is this, that however we slit the surface of a sphere, we cannot unroll the surface into a plane area, as we have done in the case of a cylinder and a cone. There is a method of dealing with the problem, but it is not easy. It is

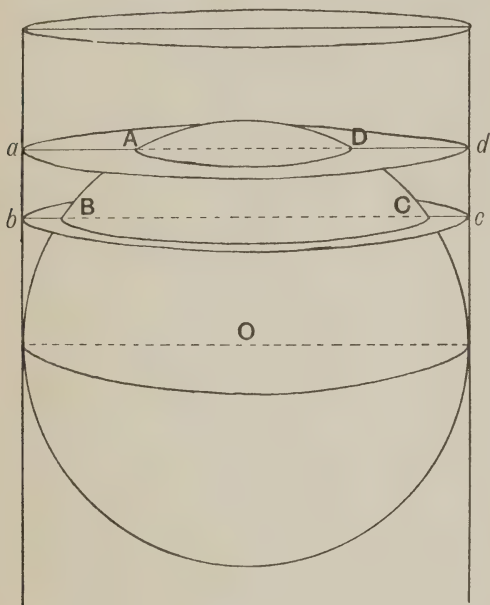


FIG. 8.

given in Appendix I. It is there shown that if a cylinder be drawn to circumscribe the sphere—*i.e.*, to touch it round the circumference of a “great circle”—and if two planes be drawn perpendicular to the axis of the cylinder, thus enclosing between them the “zone” of the sphere, shown in Fig. 8 by ADCB,

and if these planes cut the cylinder in circles, shown in perspective in the figure by ad , cb , then the area of the curved surface of the zone of the sphere ADCB is equal to the area of the curved surface of the corresponding portion of the cylinder $adcb$. If the perpendicular distance between the planes is h , and the radius of the sphere is r , this area is $2\pi rh$.

From this we can get the total surface of the sphere by putting $h=2r$. This gives $4\pi r^2$ for the area of the sphere.

In Fig. 9, O is the centre of a circle, AC is at right angles to OB. If the figure be rotated about BOB',

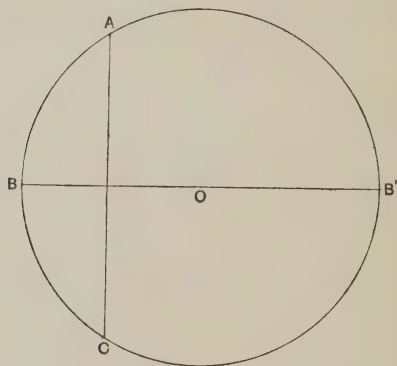


FIG. 9.

the part of the sphere represented in section by ABC is called a spherical cap. If the distance of AC from O is d , the area of the curved surface of the cap is $2\pi r(r-d)$. (See also Fig. 11.)

EXAMPLES I.

1. Find an expression for the area of an equilateral triangle of side a inches.

2. Find an expression in terms of a and x for the area of a triangle of which the sides are a , $a + \frac{x}{2}$, $a - \frac{x}{2}$.

3. By how much per cent. does the area of a triangle, the sides of which are 3·8, 4, 4·2 inches, fall short of that of an equilateral triangle of side 4 inches?

4. A rectangular area a cm. long by b cm. broad, has its corners rounded off into circular quadrants of radius r cm. Find an expression for its area.

5. Find the area in square inches of a triangle, of which the sides are 6 inches, 6·15 inches, and 5·88 inches.

What would be the side of an equilateral triangle of equal area?

6. AC is a straight line of length l , bisected at B. AD, BE, CF, are straight lines at right angles to ABC on the same side of it, of lengths h_1 , h_2 , h_3 , respectively. The straight lines, DE and EF, are joined. Show that the area of the complete figure ADEFCA is $\frac{h_1 + 2h_2 + h_3}{4} \times l$.

7. ABC is a triangle. P is a point in AB, so that AB is n times PB. PQ is drawn parallel to BC to meet AC in Q. Find the ratio of the area of the trapezium PQCB to the area of the triangle ABC.

8. An iron plate, shaped like a trapezium, has its two parallel sides, 2 feet 7 inches and 3 feet 4 inches

long respectively, and the distance between them, 1 foot 8 inches. Thirty-two equal circular holes are punched in the plate, and its weight is then found to be 17 per cent. less than before. What is the diameter of each hole?

9. Regular octagons are respectively inscribed and circumscribed to a circle of radius r . Find expressions for their areas. Verify that one-third of the area of the inscribed octagon added to two-thirds of the area of the circumscribed gives a close approximation to the area of the circle.

10. Find the total surface area of a right prism, of which the cross-section is a regular pentagon, of side 5·3 inches, and the height is 7·9 inches.

11. The base of a right pyramid is a square of side a , and the vertical height is h . Find an expression for the total area.

12. The parallel sides of a trapezium are a and b . A straight line is drawn parallel to and half-way between them, thus dividing the original figure into two smaller trapeziums. What is the ratio of the areas of these?

13. The boundary arc of a sector of a circle measures 5·2 inches, and it subtends 85° at the centre of the circle. Find the radius of the circle and the area of the sector.

14. Show that the expression for the area of a segment of a circle of radius r , bounded by an arc of length l , and by the chord joining its ends, is $\frac{1}{2}r\left(l - r \times \sin \frac{l}{r}\right)$, the angle being measured in radians.

15. If $2a$ be the length of the chord bounding a segment of a circle, and h be the "height" of the

segment (*i.e.*, the length of the straight line joining the middle points of the chord and the arc), show that the radius of the circle is $\frac{a^2+h^2}{2h}$.

Find the area of the segment in the case when $a=1\frac{1}{2}$ inches and $h=1$ inch.

16. A sector of a circle of radius r is bounded by an arc of length l . This is cut out of paper, and the edges brought together so as to form a cone. Find an expression for (i.) the total surface area of the cone (imagined solid); (ii.) the vertical angle of the cone.

17. The vertical angle of a cone is 40° , and the vertical height is 6 inches. Calculate the total surface area.

18. A conical tent has the circumference of its base 40 feet, and its vertical height is 12 feet. Find (i.) the area of ground covered; (ii.) the area of canvas required.

19. A sector of a circle is bounded by an arc subtending an angle α at the centre. It is cut out of paper, and the bounding radii brought together so as to form a cone of vertical angle θ . Find the equation connecting α and θ . What is the value of θ if the original sector were a semicircle?

20. Two concentric circles, O being the common centre, are of radii r_1 and r_2 respectively ($r_1 > r_2$). OCA, ODB are radii meeting the inner circle in C and D, and the outer in A and B. The length of the arc AB is l . Find an expression for the area ABDC.

21. Assuming that the area of a sector of a circle is $\frac{1}{2}$ arc $\times$ radius, prove that the area of the curved surface of a frustum of a cone is $\pi(R+r)L$, where R and r are the radii of the ends and L is the slant.

22. A frustum of a pyramid is 12 inches high, and its base is a square of side 15 inches, its top a square of side 5 inches. A cover to fit over the top and sides is to be in one piece, cut from a square. Draw the true shape of the cover laid out flat on a plane, and find the length of the side of the square from which it is cut. What percentage of the material is wasted?

23. A funnel, made of sheet tin, consists of a cylindrical part 4 inches long, attached to a frustum of a cone. If the total height is 8 inches, the radius of the cylindrical part $\frac{1}{4}$ inch, and the radius of the broad end $2\frac{1}{2}$ inches, find the area of tin required.

24. Show that, if h is the height, and r the radius of the base of a cap of a sphere, the curved surface can be expressed in the form $\pi(r^2 + h^2)$.

25. A hemisphere, of which the plane base is AOB, has the top sliced off and removed, the top being the part above a plane parallel to AOB, and 1.5 inches from it. The radius of the sphere is 6 inches. Calculate: (i.) the curved surface of the hemisphere as thus truncated; (ii.) the total surface of this solid.

26. A cylindrical hole, 4 inches in diameter, is cut right through the centre of a sphere of diameter 6 inches. Calculate the total surface of the piece cut out.

27. A sphere of diameter, 3 feet, represents the earth. The Northern Hemisphere is painted in three colours: from the Equator to latitude 30° , yellow; from 30 to 60° , blue; and thence to the Pole, red. Find the total surface covered by each colour respectively.

CHAPTER II

SOME SIMPLE VOLUMES

§ 6. **The Measurement of Volumes.**—Volumes are measured in cubic units, a cubic inch, cubic centimetre, etc., being the volume of a cube, each side of which is an inch, a centimetre, etc.

Just as we derived from the definition of the unit of area the fact that the area of a rectangle, whose sides measure a units and b units is ab square units, so it follows from the definition of the unit of volume that the volume of a cuboid (or rectangular parallel-piped), of which the sides measure a units, b units, and c units, is abc cubic units.

§ 7. **Right Prisms and Right Cylinders.**—If we have a right prism, the height of which is h units and the area of the base A square units, then the volume is Ah cubic units.

For if the base is a polygon of area 3.08 square inches, and the height is 4 inches, the figure forming the base can be cut up and rearranged, so as to form exactly 308 little squares, each of side one-tenth of an inch.* On each of these stands a cuboid, the volume of which is $4 \times .01$ cubic inches. Therefore, the volume of the whole prism is 308 times this, or 4×3.08 cubic inches.

* It is here assumed that the area is *exactly* 3.08 square inches.

The volume of a cylinder of radius r and height h is $\pi r^2 h$.

[*Note.* — The difficulty about incommensurable numbers crops up again here. If the radius of a cylinder is 1 inch, the area of the cross-section is π square inches. Obviously this does not mean that the area bounded by the circumference of the circle can be cut up and fitted into any whole number of small squares, so that the proof of the paragraph above does not apply. The explanation is that of the note to § 3. If the circle is drawn on very finely grained squared paper, and the number of the smallest squares made by the graining on the paper and wholly enclosed by the circle be counted, it will be found that this number, multiplied by the area of each square expressed as a decimal of a square inch, will be approximately equal to 3.14159 . . . and that the approximation becomes closer as we use more finely grained paper. The rule for the volume of a cylinder is thus shown to be true.

The rule $A \times h$ applies also to any figure of constant cross-section—*i.e.*, the cross-section may be an ellipse or any other curved figure, not necessarily rectilinear or circular.]

For an oblique prism or oblique cylinder the volume is still given by $A \times h$ cubic units, but h is now the perpendicular distance between the top and bottom. This is the analogy in solid geometry to the fact that the area of a parallelogram is equal to that of a

rectangle on the same base, and between the same parallels.

§ 8. **Right Pyramid and Right Cone.**—Just as the area of a triangle is one-half the length of the base multiplied by the vertical height, so it can be shown that the volume of a right pyramid or cone is one-third of the area of the base multiplied by the height.

The proof of this by elementary methods—*i.e.*, without using Calculus—is not easy. It can, however, be shown that a triangular prism can be cut up into three figures, two of which are pyramids (though not right pyramids), and these three can be proved equal in volume. This may be illustrated by a model.

But though the proof of the general fact is not easy, it is simple to verify that the rule gives the accurate result in a certain particular case.

O is the centre, and ABCD one face of a cube. OABCD is thus a pyramid on a square base.

Six such pyramids can be exactly fitted into the space inside the cube.

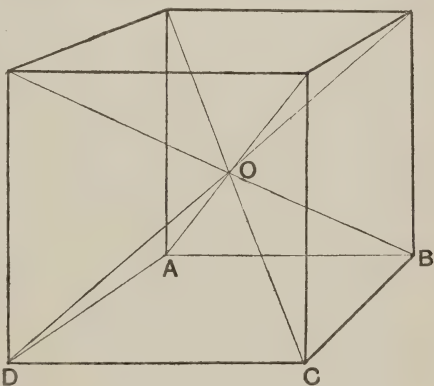


FIG. 10.

Therefore, if a be the length of the side of the cube,

the volume of OABCD will be $\frac{1}{6}a^3$. But the area of ABCD is a^2 , and the height of the pyramid is $\frac{1}{2}a$. The rule is thus verified in this case.

The rule is also true in the case of oblique pyramids and oblique cones. The only difference is that h must be measured perpendicular to the base, and this is not along the axis, as in the case of the right pyramids and cones.

§ 9. Volume of a Spherical Sector.—In the same way that the rule for the area of a triangle has been shown in § 3 to lead to the rule, one-half the length of arc multiplied by the radius, for the area of a sector of a circle, so the rule for the volume of a cone can be shown to lead to the rule, one-third the area of the surface of the cap of a sphere multiplied by the radius for the volume of a spherical sector.

The idea underlying the proof of § 3 is that of a very small arc which moves along, taking up consecutive positions from one end of the arc to the other. The area of the very small sector formed in each position is given by the triangle rule, since the arc is so short that it may be considered as straight. By adding up all these very small sectors, we get the area given by the rule.

In the same way we can imagine a very small circular wafer which moves about, taking up all imaginable positions on the spherical cap. For each of its positions the volume of the corresponding spherical sector is given by the rule for the cone—*i.e.*, one-third the radius multiplied by the area of the wafer. Thus the rule follows by addition.

The difficulty that the whole surface of the cap would not be covered by the various positions of the wafer, and that there would be trouble along the boundary, is met in the same way as indicated in the note to § 3—by imagining the wafer to be gradually reduced in size, we get closer and closer approximations to the theoretical result.

If O be the centre of a sphere, and $APCQ$ a section of which the centre is D , the area of the surface of the cap ABC is (by § 5) $2\pi r(r-d)$, where OD is of length d .

Thus the volume of the sector is $\frac{2}{3}\pi r^2 \times (r-d)$.

If we require the volume of the spherical segment, bounded by the cap ABC and the plane $AQCP$, we must subtract from the above result the volume of the cone $OAQCP$ —i.e., $\frac{1}{3} \times \pi(r^2 - d^2) \times d$.

It follows also, from the above, that

the volume of the complete sphere is $\frac{4}{3}\pi r^3$ multiplied by the area of the surface of the sphere, which gives $\frac{4}{3}\pi r^3$.

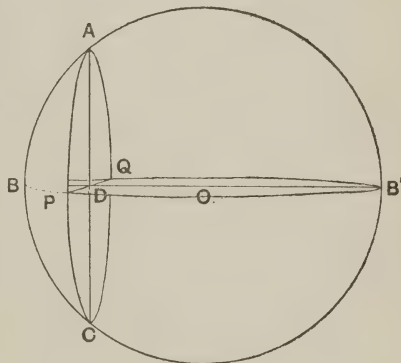


FIG. 11.

§ 10. The Theorems of Pappus.—Two interesting rules first discovered by Pappus (a Greek mathematician who lived at Alexandria towards the close of the third

century after Christ), are of some importance in their applications.

These are:

(i.) The volume of a solid traced out by the revolution of a plane figure about an axis in its plane is given by, the area of the figure multiplied by the length of the arc traced out by the centre of area (or centre of gravity) of the figure during the revolution.

(ii.) The surface traced out by the revolution of a plane line (straight or curved) about an axis in its

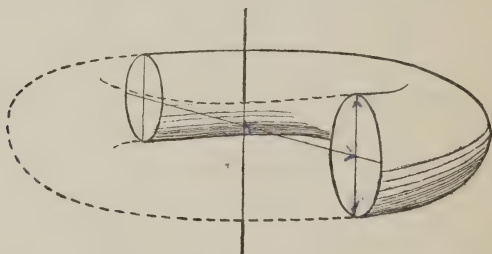


FIG. 12.

plane, is equal to the length of the line, multiplied by the length of the arc traced out by the centre of gravity of the line during its revolution.

These can be applied to give the surface area and volume of an anchor-ring. This is formed by the revolution of a circle of radius r about a straight line distant d from the centre of the circle.

By the rules given, the volume of the anchor-ring is $\pi r^2 \times 2\pi d$, or $2\pi^2 r^2 d$, and the surface area is $2\pi r \times 2\pi d$, or $4\pi^2 r d$.

EXAMPLES II.

1. A rectangular sheet of paper measures a inches by b inches. Two of the edges are brought together, and the paper is bent into the form of the surface of a cylinder. What are the volumes of the two cylinders that can be thus formed?

2. O is one corner of a cube, OA, OB, OC, the three edges meeting in O. Each of these is of length a . Find in terms of a : (i.) the volume of the pyramid OABC; (ii.) the area ABC; (iii.) the length of the perpendicular from O on to the plane ABC.

3. The radius of the sector of a circle is l , and the angle of the sector is θ . The two bounding radii are brought together, and the figure is deformed into the shape of a cone. Find the volume of the cone in terms of l and θ .

4. Extract from the rules of cricket: A cricket ball "shall weigh not less than 5 ounces, nor more than $5\frac{3}{4}$ ounces." Also "it shall measure not less than 9 inches nor more than $9\frac{1}{4}$ inches in circumference." Find out whether, with these restrictions, it was possible to make a cricket ball that would sink in fresh water.

5. A solid spherical leaden ball, d inches in diameter, is beaten out into a circular sheet of uniform thickness, t inches. Find an expression for the radius of the sheet.

6. A sector of 70° is cut away from a circular piece of material of 3 feet radius, and the remainder is made into a conical tent. Find the height of the tent and its cubic content.

7. A casting is made in the form of a cylinder capped by a cone. The diameter of the cylinder and of the base of the cone is 6 inches. The length of the cylinder is 10 inches, and the height of the cone is 4 inches. Find the weight of the casting, if the metal used weighs 0.28 lb. per cubic inch.

8. A very long strip of paper is rolled on a cylinder of wood 1 inch in diameter, and the total diameter is 2 inches. Find what would be the total diameter of the new roll if the same strip were coiled on a cylinder of wood 2 inches in diameter.

9. The height of a solid cone is $\frac{3}{5}$ of its diameter. The total surface is equal to the total surface of a cylinder of the same diameter. Find the ratio of their volumes.

10. A bottle is made of glass $\frac{1}{8}$ inch thick, and both body and neck are cylindrical. The outside dimensions are as follows:

Body 2.0 inches diameter, 2.2 inches high.

Neck 1.3 inches diameter, 0.9 inch high.

Find how far down the stopper must go in order that the bottle may hold exactly 5 cubic inches.

11. Find the volume of a buoy consisting of a cone joined to a hemisphere, the height of the cone and the radius of the hemisphere being each 2 feet.

12. A hollow vessel in the form of an inverted cone, having a circular top of diameter 1 foot and a depth of 1 foot, is filled with water. Find in cubic inches the volume of water contained in the vessel.

A long cylinder, 5 inches in diameter, is slowly

lowered into the vessel, and allowed to rest in contact with the curved surface. Find the volume of water which overflows.

13. The cross-section of a steel bar is a regular five-sided figure of side s inches. The weight of the steel is 0.28 lb. per cubic inch. Find an expression for the weight in pounds per foot length, in the form $a \times s^2$, giving the value of the constant a correct to three significant figures

14. The height of a frustum of a cone is h , and the radii of the top and bottom are r_1 and r_2 respectively. Prove that the volume is $\frac{1}{3}\pi h(r_1^2 + r_1r_2 + r_2^2)$.

15. The height of a frustum of a right cone is h , the areas of the top, the bottom, and of a cross-section midway between the top and bottom are A_1 , A_2 , A_3 respectively. Show that the volume of the frustum is $\frac{1}{6}h(A_1 + A_2 + 4A_3)$, which can be transformed to $\frac{1}{3}h(A_1 + A_2 + \sqrt{A_1A_2})$.

16. An obelisk of concrete is made in the form of a frustum of a pyramid, on a square base capped by a smaller pyramid. The edges of the top and bottom faces of the frustum are 5 feet and 10 feet, the height being 4 feet. The top pyramid is 1 foot high, standing, of course, on a base of edge 5 feet. How many cubic feet of concrete will be required to construct it?

17. A wooden pole used as a flagstaff is 60 feet tall. It tapers gradually from a diameter of 14 inches at the bottom to one of 4 inches at the top, the shape being that of a frustum of a cone. Calculate its volume in cubic feet.

18. A street lamp has its top and bottom flat and

square. Each of its slanting faces is a trapezium, with sides 5 inches, 10 inches, 5 inches, 8 inches long. Find the volume of air contained in it.

19. A hole $\frac{3}{4}$ -inch diameter is drilled right through a solid cone, base 3 inches diameter and 4 inches high. The axis of the hole is the same as that of the cone. Find the volume of the remaining solid.

20. A tumbler is in the form of a frustum of a cone. The diameters of top and bottom are respectively 3 inches and 2 inches, and the depth is 5 inches (all inside measurements). Express as the decimal of a pint the amount of liquid the tumbler will hold (1 gallon=277·3 cubic inches).

21. If S is the total surface area, and V the volume of a cylinder of radius r and length l , find a formula for $\frac{S}{V}$ in terms of r and l .

Find the total surface area of a charge of 21 lb. of cordite, made up into threads of diameter 0·04 inch and length 8 inches, assuming that a cubic foot of cordite weighs 96 lb.

22. A cone is inscribed to a sphere, the ratio of the height of the cone to the diameter of the sphere being c . Prove that the ratio of the volume of the cone to the volume of the sphere is $2c^2(1-c)$.

23. A spherical segment of height h is cut from a sphere of radius R . The radius of the plane face bounding the segment is r . Prove that the volume of the segment can be expressed in either of the forms: (i.) $\frac{1}{6}\pi h(3r^2+h^2)$; (ii.) $\frac{1}{3}\pi h^2(3R-h)$.

24. A solid figure consists of a cone and a spherical

segment standing on opposite sides of a common circular base of radius r . If the total height of the figure is H , and the height of the segment is h , find an expression in terms of r , H , and h for the volume of the figure.

25. A buoy consists of a spherical segment joined to a cone. The height of the cone is 12 feet, and the radius of the common base of the cone and spherical segment is 5 feet. The total height of the buoy is $19\frac{1}{2}$ feet. Find the volume in cubic feet.

26. A spherical ball, 6 inches in diameter, is placed in a conical funnel, 9 inches across the mouth and 6 inches deep. Find the air-space under the ball.

27. If a sphere is cut by two parallel planes, show that the volume of the slice so formed is $\pi t \left(R^2 - d^2 - \frac{t^2}{12} \right)$, where t is the thickness of the slice (*i.e.*, the distance between the planes), R is the radius of the sphere, and d is the distance from the centre of the plane parallel to, and halfway between, the two bounding planes.

If one of the bounding planes pass through the centre of the sphere, show that the volume of the slice is $\pi t \left(R^2 - \frac{t^2}{3} \right)$.

28. For a bowl of height h , in the shape of a segment of a sphere of radius R , the volume is given by the formula $\pi h^2 \left(R - \frac{h}{3} \right)$. If the height of the bowl is 2.31 inches, and the volume is 60 cubic inches, find the radius of the sphere of which the bowl is a segment.

29. A cable consists of 200 links of steel, 2 inches

diameter in section, worked into circular rings of 6 inches outside diameter. Find the volume in cubic inches of one link. Find the weight, in tons, of the cable, the steel being of sp. g. 7.85.

30. Show that Pappus' first theorem can be applied to give the volumes of (i.) a hollow cylinder; (ii.) a cone.

31. By Pappus' theorems, calculate the distances from the centre of the centres of gravity of (i.) the area of a semicircle; (ii.) the arc of a semicircle.

32. A triangle ABC rotates through 360° about the side AB. Show that the volume so traced out is the same as that of a sphere of radius

$$\sqrt[3]{\frac{s(s-a)(s-b)(s-c)}{c}}.$$

where s is the semiperimeter of the triangle.

33. ABCD is a trapezium, the sides DA and CB being parallel, and the angles at A and B right angles. AD=4 inches, AB=6 inches, BC=8 inches. Find the volume of the solids formed by rotating the figure about (i.) AB, (ii.) BC. Hence, using Pappus' theorem, find the position of the centre of gravity of the trapezium—*i.e.*, its distances from AB and BC.

34. Use Pappus' second theorem to show that the area of the curved surface of a frustum of a cone, the slant (of the frustum) being L and the radii of top and bottom R and r , is given by the formula $\pi L(R+r)$.

35. A cylindrical hole is bored through a sphere, the axis of the cylinder being a diameter of the sphere.

(Although the hole is called cylindrical, it really consists, of course, of a cylinder, together with two spherical segments.) The length of the cylinder is $2z$. Show that the volume remaining is $\frac{4}{3}\pi z^3$, and is thus independent of the radius of the sphere.

36. If a spherical cap be circumscribed to a cone, height h , slant l , then (i) the area of the curved surface of the cap is πl^2 ; (ii) the volume contained between the cone and the cap is $\frac{1}{6}\pi l^2 h$.

(Cf. Examples I. 24, and II. 23.)

CHAPTER III

APPROXIMATE METHODS FOR DETERMINING THE AREAS OF CURVES

§ 11. It frequently happens, in applying mathematics to practical problems, that we need to determine the area enclosed by a curve, or the area between a curve and a straight line, or the area bounded by a curve and three straight lines.

For instance, the problem may be to determine the area of the cross-section of a stream. Soundings have been taken at various distances from one bank across the stream, and a curve showing the shape of the cross-section can thus be drawn. We need to approximate to the area contained between this curve and the straight line representing the horizontal surface of the water.

Still more frequently, in connection with problems in applied mechanics, we have a curve drawn out on squared paper, and we have to determine, approximately, the area contained between this curve, the axis of x , and the ordinates drawn perpendicular to this axis from the two points at the ends of a certain arc of this curve.

For instance, Fig. 13 represents, on a reduced scale, the graph of the parabola $y=x^2$ drawn from the origin up to the point (2, 4). Units, 1 inch for both x and y .

Suppose we require the area contained between the arc OP, the axis OM (the base-line), and the ordinate PM.

We can get an approximation to this by counting squares. Suppose that we count up the little squares, the side of each of which is $\frac{1}{16}$ inch. We find that 238 such squares are entirely enclosed, and that the curve cuts through 57 more such squares.

(The reader should verify this). If we

reckon, as one square, each of those of which more than one-half appears to be enclosed within the area sought, and neglect those of which less than one-half seems to be enclosed, we add 30 more squares. This gives 268—i.e., 2.68 sq. in. as an approximation to the area

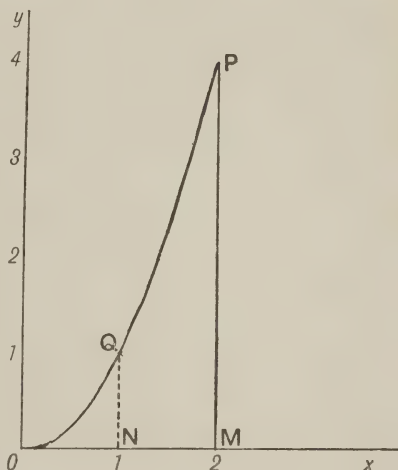


FIG. 13.

sought.

We can check this result by theory. For there is an "area-rule" for parabolas, just as there is an area-rule for circles. The area-rule for the area under a parabola* can be shown by Calculus methods to be:

* By "parabola" is here implied a parabola whose axis is at right angles to the base-line.

Divide the length of the base-line by 6 and multiply by the sum of the extreme ordinates and four times the middle ordinate.

In the case considered, the extreme ordinates are 0 and 4, and the middle ordinate (QN) is 1. The length of the base-line is 2. So the theoretical area is $\frac{1}{6}$ or $2\frac{2}{3}$ sq. in. What this theoretical area really means was explained for the case of the circle in § 3. If we took more and more finely grained squared paper and continued the square-counting process, we should approximate more and more closely to this result.

EXAMPLES IIIa.

1. Draw the portion of the curve $y = \frac{10}{x}$ between the points (2, 5) and (5, 2). Use squared paper graduated in inches and tenths, taking 1 inch as unit for both x and y . By counting squares, as described in the above paragraph, estimate the area in square inches contained between the curve, the x axis, and the ordinates at the two given points.

2. By the same process as that described in question 1, find the area contained between the curve $y = 10 \log x$, the x axis, the point (1, 0), and the ordinate at the point (3, 4.771).

§ 12. But this method of counting squares is very tedious. Let us enquire, therefore, if we can carry out the approximation in a more convenient way. Before doing so we will define one or two terms.

AC is an arc of a curve, BD a straight base-line, AB, CD are the bounding ordinates drawn from the ends of the arc to the base-line. We shall speak of the area ABDC as the area "under the curve"—*i.e.*, under the arc AC

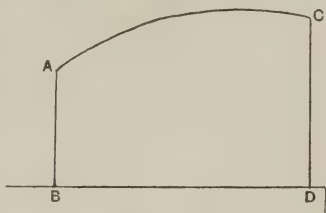


FIG. 14.

The "mean (or average) height" or "mean ordinate" of such a curve is

an ordinate whose length is that of the second side of a rectangle, one side of which is the base-line BD, and the area of which is the area under the curve.

Thus, if AC be a straight line, the mean height is half the sum of AB and CD (Fig. 15).

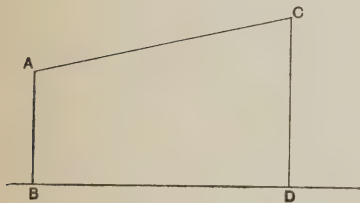


FIG. 15.

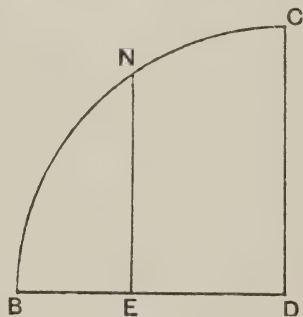


FIG. 16.

The mean height of the quadrantal arc of a circle BC, radius a is:

$$\frac{\text{area BCD}}{\text{base-line BD}} = \frac{\frac{\pi}{4} a^2}{a} = 0.7854a. \quad (\text{Fig. 16})$$

Notice that in this case NE, the ordinate of the curve which is equal in length to the mean height, is not half-way between B and CD, but nearer to B.

In Fig. 17, AC is the arc of a parabola of which the equation is $y=0.3x^2-0.1x+1$. The curve thus passes through the points (0, 1), (1, 1.2), (2, 2). By

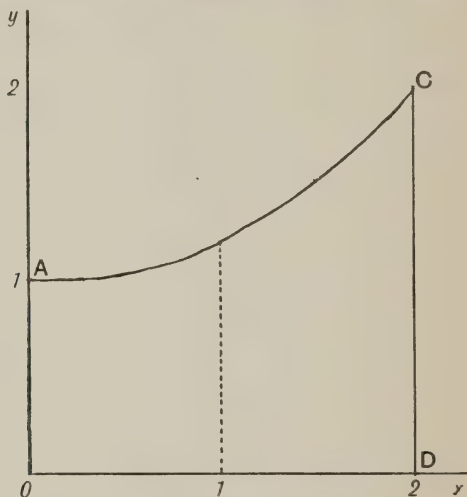


FIG. 17.

the area-rule, given in § 11, for such an arc of a parabola, the area will be $\frac{1}{6} \times 2 \times (1+4.8+2)$, and the mean height will be $\frac{7.8}{6}=1.3$.

From the definition it follows that, in all cases, the number of units of length in the mean height, multiplied by the number of units of length in the base-line, gives the number of units of area in the space "under the curve."

EXAMPLES IIIb.

1. In Fig. 16 calculate (each in terms of a) the lengths BE and ED.

2. What are the mean heights of the portions of the parabola $y=x^2$ lying between (i.) $(0, 0)$ and $(1, 1)$;

(ii.) $(-1, 1)$ and $(1, 1)$? (The x axis is the base-line.)

3. What is the mean height of that portion of the parabola $y=x^2-2x+3$, lying between the points $(1, 2)$ and $(5, 18)$? (The x axis is the base-line.)

4. AC is a straight line of length 6 inches, bisected at B. AD, BE, CF, are straight lines at right angles to ABC on the same side of it, of lengths 0.3 inch, 3.7 inches, 9.2 inches respectively. What is the mean height of the area under the broken straight line DEF? (Cf. Examples I., No. 6.)

5. If a parabola be drawn with its axis vertical, show that the area cut off by any horizontal line is $\frac{2}{3}$ that of the circumscribing rectangle. (See footnote on p. 31.)

6. Show that the area contained between the parabola whose equation is $y=kx^2$, the x axis, and the ordinates at the points where $x=a$ and $x=b$ is $\frac{1}{3}k(a^3-b^3)$.

7. PQRS is a trapezium in which the angles PQR, QRS, are right angles. It is required to draw a parabola to pass through P and S, whose axis shall be perpendicular to QR, and which shall divide the trapezium into two parts of equal area. Show that if L, M, are the mid-points of PS, QR, respectively, the parabola will cut LM at a point N, such that $MN=\frac{1}{4}LM$.

8. Q_1, Q_2, Q_3, Q_4, Q_5 , are five points on the x axis,

such that $Q_1Q_2=Q_2Q_3=Q_3Q_4=Q_4Q_5=d$. A parabola is drawn where axis is parallel to the y axis, and ordinates $P_1Q_1, P_2Q_2, P_3Q_3, P_4Q_4, P_5Q_5$, are drawn to meet the curve in P_1, P_2, P_3, P_4, P_5 , respectively.

(i.) Calling these ordinates h_1, h_2, h_3, h_4, h_5 , use the parabola-rule to express the area under the arc P_1P_5 , in terms of h_1, h_3, h_5 , and d .

(ii.) By treating the two parts of the arc P_1P_3 and P_3P_5 separately, show that this same area is also equal to $\frac{d}{3}(h_1+4h_2+2h_3+4h_4+h_5)$.

(iii.) Thence express h_3 in terms of h_1, h_2, h_4, h_5 , and show that another expression for the area is

$$\frac{2d}{9}(h_1+8h_2+8h_4+h_5).$$

§ 13. There are various methods of finding the approximate area under a curve, without having recourse to

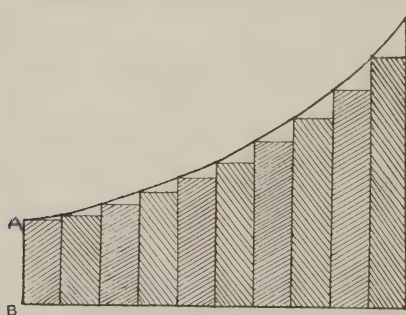


FIG. 18.

the tedious counting of squares. These methods all reduce themselves to the same principle—that of modifying the curve, so as to bring it into a form in which the area can be estimated by one

of the simple rules we have learnt.

Consider the curve AC shown in Figs. 18 and 19.

In each case we have divided the base-line BD into 10 equal parts, and we have drawn the ordinates at the points of division. Counting the first and last, we have thus 11 ordinates in all.

Denote the lengths of these by h_1, h_2 up to h_{11} . Thus, $AB=h_1$, and $CD=h_{11}$.

In Fig. 18 we have completed rectangles by drawing short lines parallel to the base-line, through the tops of ordinates 1 to 10.

In Fig. 19 we have done the same thing through the tops of ordinates 2 to 11.

We will speak of Fig. 18 as the "inner staircase," and of Fig. 19 as the "outer staircase." Let $BD=l$.

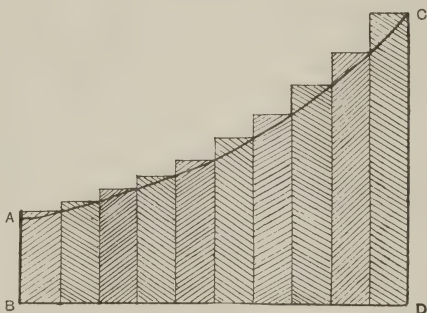


FIG. 19.

The sum of the areas of the shaded rectangles, under the inner staircase, is:

$$\begin{aligned}
 & \frac{l}{10} \times h_1 + \frac{l}{10} \times h_2 + \dots + \frac{l}{10} \times h_{10} \\
 &= l \times \frac{h_1 + h_2 + \dots + h_{10}}{10} \\
 &= l \times \text{the arithmetic mean of the ordinates} \\
 & \quad h_1, h_2, \dots, h_{10}.
 \end{aligned}$$

Now, the sum of the areas of these shaded rectangles is less than the actual area required, by the sum of the

unshaded triangular patches. The sum of these will not amount to very much, so that we can say that l , multiplied by the arithmetic mean of the ordinates $h_1, h_2, \dots, h_{10}$, is approximately equal to, though less than, the area under the curve AC.

Thus, the arithmetic mean of the ordinates $h_1, h_2, \dots, h_{10}$ is an approximation for the mean ordinate of the curve AC.

Similarly, if we had taken the outer staircase of Fig. 19, the sum of the shaded rectangles would have been rather in excess of the real area, but an approximation to it.

Thus, the arithmetic mean of the ordinates $h_2, h_3, \dots, h_{11}$, is an approximation for the mean ordinate of the curve AC.

Whereas the previous approximation was in defect, this one is in excess of the value of the mean height sought.

It might be suggested that we should take all 11 ordinates and treat the arithmetic mean of these as an approximation to the mean ordinate sought. This would seem at first sight to be reasonable, for it would increase our first result, which was admittedly in defect (for, from the figure, since h_{11} is greater than any of the preceding ordinates, so $\frac{h_1+h_2+\dots+h_{11}}{11}$ must be greater than $\frac{h_1+h_2+\dots+h_{10}}{10}$).

But, in the first place, this is simply a "shot in the dark." It cannot be explained why $\frac{l}{11} \times (h_1+h_2+\dots+h_{11})$

. . . h_{11}) should give a close approximation to the area under the curve, except on general, rather vague, grounds. The expression does not represent the sum of a series of rectangles closely following the curve in the same way that either of the previous expressions does.

In the second place, the only obvious reason why we should get a better result by averaging all eleven ordinates in this way is that the curve drawn in Figs. 18 and 19 is of the shape there shown, sloping upwards to the right. If the curve had turned somewhere between A and C and had begun to descend, we could not tell whether it would have improved the result or made it worse, if we had taken the arithmetic mean of all eleven ordinates as an approximation to the mean height, instead of taking 1 to 10 inclusive, or 2 to 11 inclusive.

If, for instance, the curve were of the shape shown in Fig. 20, we can no longer speak of the "inner" and "outer" staircases.

The staircases, shown in full and dotted lines, cross. Thus, we cannot say whether the sum of the rectangles under the first staircase is in

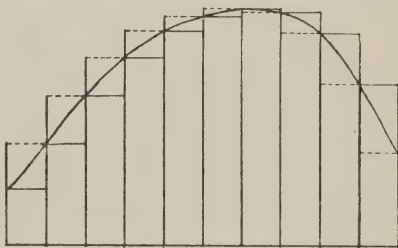


FIG. 20.

excess or defect of the area under the curve. But it is clear that each of the rectangle-sums under the staircases will be a close approximation to the area; in other words, that $\frac{1}{10}(h_1 + h_2 + \dots + h_{10})$ and

$\frac{1}{10}(h_2+h_3+\dots h_{11})$ will each be a close approximation to the mean ordinate of the arc AC.

It is not worth considering whether $\frac{1}{11}(h_1+h_2+\dots h_{11})$ would, or would not, in any particular case, give a better or a worse result than either of the others. It would be "somewhere near" each of these others.

We should get improved results by making the stairs narrower—*i.e.*, by taking more of them.

We have dwelt on this method of approximating to the area under a curve not because it is a method generally employed in practice, but because of its importance in the theory of the application of the area under a curve to certain problems in mechanics. We will now pass on to the practical methods which are employed to measure such areas.

§ 14. **The Mid-Ordinate Rule.**—In this case we take a staircase whose stairs are midway between those of the staircases previously considered.

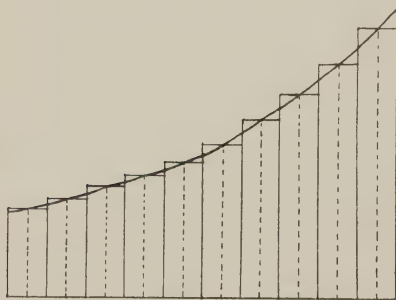


FIG. 21.

Divide the base-line into a convenient number of equal parts, say ten as before, and draw the ordinates at the middle points of each of these divisions. This gives us ten middle-ordinates, which we will denote by $m_1, m_2, m_3 \dots m_{10}$. Draw the staircase through the

tops of these middle-ordinates. It is clear that the area under this staircase will be a close approximation to the area under the curve.

This area is $(m_1 + m_2 + m_3 + \dots + m_{10}) \times \frac{l}{10}$.

Thus, the arithmetic mean of the ten mid-ordinates will be a close approximation to the mean-ordinate of the curve.

This rule is known as the **Mid-Ordinate**, or the **Engineers' rule**.

We can test its accuracy by applying it to the case of a quadrant of a circle. The radius (Fig. 22) is

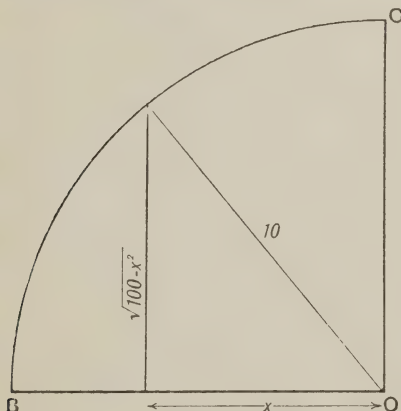


FIG. 22.

supposed to be 10 cm. If we take ten divisions to the base BO, the middle-ordinates will be at distances 0.5, 1.5, 2.5, . . . 9.5 cm. from the centre O.

Now, the ordinate at x cm. from O is, by the theorem of Pythagoras, of length $\sqrt{100 - x^2}$ cm.

The work can, therefore, be set out as follows:

<i>Distances of Mid-Ordinates from D (x).</i>	<i>Lengths of Mid-Ordinates $\sqrt{100 - x^2}$.</i>
9.5	3.122
8.5	5.268
7.5	6.615
6.5	7.599
5.5	8.352
4.5	8.930
3.5	9.368
2.5	9.683
1.5	9.887
0.5	9.987
	Sum=78.811

Arithmetic mean of mid-ordinates=7.88 (1) cm.

Accurate answer (see § 12)=7.854 cm.

So that our approximate result is in error by less than 0.4 per cent.

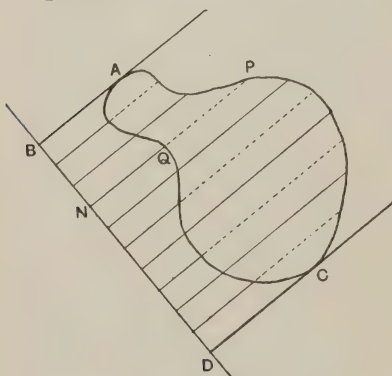


FIG. 23.

If we have to find the area of a closed curve, such as that shown in Fig. 23, we proceed as follows:

Draw two parallel tangent-lines to the curve, so as to enclose it completely. Draw the base-line BD at right angles to these and well away from the figure. Divide the

base-line into any convenient number of equal parts (5 in the figure), and draw middle-ordinates as shown.

The area under the top portion APC is the length of BD multiplied by the arithmetic mean of the five middle-ordinates, such as NP.

The area under the lower portion AQC is the length of BD multiplied by the arithmetic mean of the five middle-ordinates, such as NQ.

Thus the two areas can be found, and their difference gives the area enclosed by the curve.

It is, however, simpler to measure the five middle-breadths, such as PQ. The arithmetic mean of these gives the mean breadth of the figure. This, multiplied by the length of BD, gives the area sought.

EXAMPLES IIIc.

1. Calculate the ordinates of the curve $y = \frac{x^2}{10}$, at the points at which $x = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, \dots, 7\frac{1}{2}$. Hence, using mid-ordinate rule, find the area in square units under the curve between the origin and the point (8, 6.4). Compare the answer with the "correct" result, as obtained by the rule for a parabola given in § 11.

2. Using four-figure tables, calculate the ordinates of the curve $y = \frac{10}{x}$ at the points at which $x = 2\frac{1}{4}, 2\frac{3}{4}, 3\frac{1}{4}, 3\frac{3}{4}, 4\frac{1}{4}, 4\frac{3}{4}$. Hence, using mid-ordinate rule, find the area in square inches under the curve between the points (2, 5) and (5, 2). Scale 1 inch to a unit for both x and y . (Cf. Examples IIIa, No. 1.)

3. Use four-figure tables to find the values of $10 \log x$ for $x = 1.1, 1.3, 1.5, 1.7, 1.9, 2.1, 2.3, 2.5, 2.7, 2.9$.

Hence, find by mid-ordinate rule the area described in Examples IIIa, No. 2.

4. The rudder plate of a certain merchant ship is 12 feet high. To measure the area of the plate by the mid-ordinate rule, the height is divided into ten equal parts, and the breadth of the plate is measured at the middle of each of these equal parts. If these measurements are found to be 0.65, 1.65, 2.25, 2.6, 2.75, 2.78, 2.78, 2.75, 2.5, and 1.7 feet, calculate the area of the plate.

5. The following table gives the half-widths (h) of a horizontal section of a ship at different distances (s) from one end.

s feet	0	20	40	80	120	160	180	200	220	240
h „	0.3	7.2	12.3	18.4	19.0	17.1	14.8	10.0	4.8	0.4

Plot h against s , choosing suitable scales. By reading off mid-ordinates, calculate the area of the section (*i.e.*, twice the area under the curve indicated).

6. Two drawing pins are stuck into a board 2.5 inches apart; a loop of string, of total length 6 inches, is placed round them. The point of a pencil is then inserted into the loop, and, keeping the string taut, a closed curve, called an ellipse, is traced out. Carry out this construction, and find the area of this ellipse by mid-ordinate rule, using seven mid-ordinates.

7. Find the area in square inches of the closed figure on the folding sheet after p. 84. Trace the figure through on to your paper, and use mid-ordinate rule.

8. Plot the curve $y = \frac{1}{2} x(2-x)(5-x)$ from $x=0$ to $x=5$, taking 1 inch as unit for both x and y . Use mid-ordinate rule to find the area of each of the portions contained between the curve and the axis.

§ 15. **The Trapezium Rule.**—In this rule we give up the “staircase” idea, and attack the problem from another standpoint.

We have divided the base-line (Fig. 24) into ten equal parts as before, and drawn the eleven ordinates. Join the tops of these ordinates by straight lines. We get ten trapeziums, and the sum of the areas of

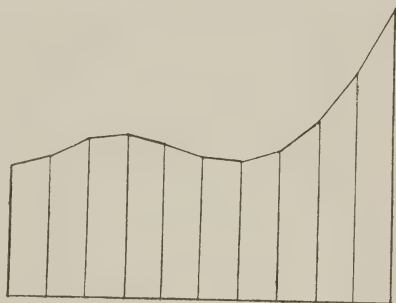


FIG. 24.

these will be a close approximation to the area under the curve. (The curve is not shown in the figure.)

The area of the first trapezium is, by the area-rule,

$$\frac{l}{10} \times \frac{h_1 + h_2}{2}.$$

So the sum of the areas is—

$$\begin{aligned} & \frac{l}{10} \left[\frac{h_1 + h_2}{2} + \frac{h_2 + h_3}{2} + \dots + \frac{h_{10} + h_{11}}{2} \right] \\ \text{i.e., } & \frac{l}{20} [h_1 + 2h_2 + 2h_3 + \dots + 2h_{10} + h_{11}] \\ \text{i.e., } & l \times \frac{h_1 + 2h_2 + 2h_3 + \dots + 2h_{10} + h_{11}}{20}. \end{aligned}$$

It follows, therefore, that the mean-ordinate as

previously defined is, approximately, given by $\frac{h_1 + 2h_2 + 2h_3 + \dots + 2h_{10} + h_{11}}{20}$, *i.e.*, the arithmetic mean of one ordinate, h_1 , two equal to h_2 , two equal to h_3 , . . . two equal to h_{10} , and one, h_{11} . (See Fig. 25 on the opposite page.)

We will test this rule by applying it to the same case as in § 14—*i.e.*, the quadrant of a circle. The ordinates can be calculated from $\sqrt{100 - x^2}$, and the work may be conveniently set out as under:

<i>Distances from O x.</i>	<i>Lengths of ordinate. $\sqrt{100 - x^2}$.</i>	<i>Multipliers.</i>	<i>Products.</i>
10	0	1	0
9	4.359	2	8.718
8	6	2	12
7	7.141	2	14.282
6	8	2	16
5	8.66	2	17.32
4	9.165	2	18.33
3	9.539	2	19.078
2	9.798	2	19.596
1	9.950	2	19.90
0	10	1	10
		Total 20	Sum 155.224

Approximate mean ordinate = $\frac{155.224}{20} = 7.76(1)$ cm.

Approximate area = 77.6 sq. cm.

(Accurate answer = 78.54 sq. cm.)

We must be careful in dealing with the area under a curve which begins and ends on the base-line (the cross-section of a stream, for instance).

In this case, the first and last ordinates are each 0. In fact, in forming such a table as that above, we must be careful to have ordinates in number one more than the number of parts into which the base-line has been divided.



FIG. 25.

EXAMPLES III*d*.

1. The water plane of a battleship is 400 feet long, and measurements of the breadth at equal distances apart (40 feet) are found to be 0, 30, 52·5, 68, 75, 74, 66, 57·5, 42, 21, 0 feet.

Calculate, by the trapezium rule, the area of the water plane.

2. A stream is 6 yards broad, its depth being 1 foot at each bank, and the depths at distances 1, 2, 3, 4, and 5 yards from one bank are 1·6, 3·2, 4·1, 4·0, and 2·2 feet respectively.

Find, by the trapezium rule, the cross-sectional area of the stream, and thence determine the flow in gallons per hour, the mean velocity of the stream being 3 miles per hour.

3. A river is 240 feet wide, and is bounded on one side by a vertical wall 19 feet deep. Soundings taken 20, 40, 60, etc., feet away from the wall give depths in feet of 19.4, 20.2, 18, 17.5, 14.9, 11, 7.2, 5.1, 4.4, 2.4, 1.1, 0. If the current is at the speed of 1 mile per hour, find, by the trapezium rule, how many cubic feet of water pass per minute.

4. Calculate the ordinates of the curve $y = \frac{x^2}{10}$ at the points at which $x=0, 1, 2, 3, 4, 5, 6, 7, 8$. Hence, using the trapezium rule, find the area in square units under the curve between the origin and the point (8, 6.4). (*Cf.* Examples IIIc, No. 1.)

5. Using four-figure tables, calculate the ordinates of the curve $y = \frac{10}{x}$, at the points at which $x=2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}, 5$. Hence, using trapezium rule, find the area in square inches under the curve between the points (2, 5) and (5, 2). Scale 1 inch to a unit for both x and y . (*Cf.* Examples IIIa, No. 1; and Examples IIIc, No. 2.)

6. Use four-figure tables to find the values of $10 \log x$ for $x=1, 1.5, 2, 2.5, 3$. Hence, find by the trapezium rule the area described in Examples IIIa, No. 2. (*Cf.* also Examples IIIc, No. 3.)

7. Find the area of the closed figure on the folding sheet after p. 84 by the trapezium rule. (*Cf.* Examples IIIc, No. 7.)

§ 16. **Simpson's Rule.**—This rule is a modification of, and an improvement upon, the trapezium rule. Any

equation of the type $y=ax^2+bx+c$ represents a parabola, of which the axis is parallel to the y axis. In this equation, a , b , and c , stand for numerical co-efficients. Thus it was shown in § 12, p. 34, that if $a=0.3$, $b=-0.1$, and $c=1$, the parabola so represented passes through the points $(0, 1)$, $(1, 1.2)$, $(2, 2)$.

The area under an arc of such a parabola is there shown to be obtainable as follows:

Bisect the base-line, so as to have three equidistant ordinates. The area is given by:

Length of base-line, multiplied by one-sixth of the sum of the two extreme ordinates and four times the middle ordinate. If we denote *half* the base-line by d , and the ordinates, as usual, by h_1 , h_2 , h_3 , the area is $\frac{d}{3} (h_1+4h_2+h_3)$.

Suppose we take such a curve as that of Fig. 18, and replace the bit of the curve joining the tops of the ordinates h_1 , h_2 , h_3 , by an arc of such a parabola. The parabolic arc will very nearly coincide with the original curve. The area under this parabolic arc is given by the formula above.

In the same way we can replace the bit of the curve joining the tops of the third, fourth, and fifth ordinates by another such parabolic arc. The area under this will be $\frac{d}{3} (h_3+4h_4+h_5)$.

We can proceed in this way up to the end of the curve bounded by h_{11} (notice that this is possible only if the total number of ordinates is odd).

Thus we get the following approximation to the area under the curve:

$$\begin{aligned} & \frac{d}{3} (h_1 + 4h_2 + h_3 + h_3 + 4h_4 + h_5 \dots + h_9 + 4h_{10} + h_{11}) \\ &= \frac{d}{3} (h_1 + 4h_2 + 2h_3 + 4h_4 + 2h_5 + \dots + 4h_{10} + h_{11}). \end{aligned}$$

[Cf. Examples IIIb, No. 8 (ii).]

In words, this may be expressed: The product of one-third of d , and (first plus last ordinate, plus twice the sum of the other odd ordinates, plus four times the sum of the even ordinates).

In the above, it must be noticed that d is the common distance between the ordinates—*i.e.*, if eleven ordinates are taken, $d = \frac{l}{10}$. Simpson's rule may also be ex-

pressed, just as the previous two rules have been expressed, as a rule to find the mean ordinate of the

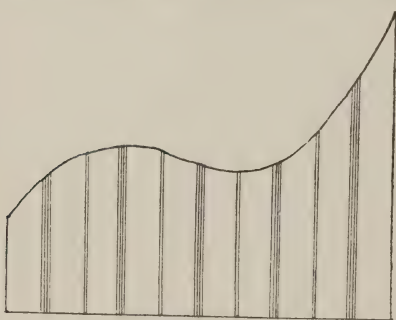


FIG. 26.

curve. It may be thus considered:

Take one ordinate equal to h_1 , four equal to h_2 , two equal to h_3 , four equal to h_4 , two equal to h_5 , and so on . . . four equal to h_{10} , one equal to h_{11} .

The arithmetic mean of these is an approximation to the mean ordinate of the curve.

The work can be set out by a method analogous

to that shown in § 15. We will test the rule by applying it to the same example as before, the quadrant of a circle of radius 10 cm.

(Compare with the work on p. 46.)

<i>Ordinates.</i>	<i>Multipliers.</i>	<i>Products.</i>
0	1	0
4.359	4	17.436
6	2	12
7.141	4	28.564
8	2	16
8.66	4	34.64
9.165	2	18.33
9.539	4	38.156
9.798	2	19.596
9.95	4	39.80
10	1	10
Total 30		Sum 234.522

Approximate mean ordinate = $\frac{234.522}{30} = 7.82$ cm.

Approximation for area required = 78.2 square cm.

(Accurate answer = 78.54 square cm.)

Notice that Simpson's rule can be applied only when the base-line has been divided into an even number of equal parts (so as to produce an odd number of ordinates), whereas the mid-ordinate and trapezium rules can be applied whether the number of equal divisions is odd or even.

With regard to the theory underlying this rule, it should be pointed out that values of a , b , and c , can be found so that a parabola, whose equation is $y = ax^2 + bx + c$, will pass through *any* three assigned

points. It may happen that one or more of these three constants will be zero. If $a=0$, the "parabola" becomes a straight line, but the area-rule for the parabola applies equally well to the straight line.

EXAMPLES IIIe.

1. The depth of the water across a river 144 feet wide is taken at points each 12 feet apart. These depths are as follows: 0, 2, 5.2, 7.3, 8.1, 10, 10.5, 12, 14, 16.5, 16, 8, 0 feet. Use Simpson's rule to calculate the area of the water section in square feet.

2. The half-ordinates of the load water plane of a vessel are 12 feet apart, and their lengths are as follows: 0, 6.0, 10.4, 14.6, 17.8, 18.5, 18.2, 17.2, 13.4, 6.3, 0 feet respectively. The total length of the plane, perpendicular to these ordinates, is 120 feet.

Calculate (a) the total area of the plane in square feet; (b) the tons per inch immersion (1 cubic foot of sea-water weighs 64 lbs.).

The seven examples in IIIId should now be worked through again, but using Simpson's rule in place of the trapezium rule.

§ 17. It remains to consider which of the above-mentioned rules should be employed in any particular case.

In the matter of accuracy, there is not much to choose for the same number of base-divisions, though, in general, Simpson's rule is most trustworthy. Though it is not advisable to have any hard and fast rules, the following general principles are suggested:

(i.) If the curve is drawn out, or has to be drawn out owing to the data being inconvenient for other treatment, or for the purpose of "smoothing" the curve, use the mid-ordinate rule.

(ii.) If ordinates are given corresponding to equal subdivisions of the base-line, and these ordinates are sufficiently reliable to obviate the necessity of drawing out the curve, Simpson's rule or the trapezium rule may be used directly, according as the number of divisions is even or odd.

(iii.) If the curve is given by an (x, y) equation, and equidistant ordinates can be simply calculated, use whichever rule is most convenient. Otherwise, calculate a few convenient points on the curve, sketch it, and use mid-ordinate rule.

[*Note as to Units.*—In plotting a curve from an (x, y) equation, it sometimes happens that we find it convenient to choose different scales for x and y . It may not at first sight be clear how this change of scale affects the area, *i.e.* in what units the calculated area under the curve, so drawn, is expressed. Proceeding by any of the above methods, this difficulty does not arise. For the mean-ordinate is determined in ins. or cms. in accordance with the scale chosen for y . The base-line is similarly known from the x scale. Thus the area can be expressed in sq. ins. or sq. cms.]

EXAMPLES IIIf.

1. Draw on squared paper the graph of $y = \sin x$, for values of x from 0 to 180° . (Scales: 1 inch to

30° for x , 5 inches to a unit for y .) Find in square inches (i.) by mid-ordinate rule, (ii.) by the trapezium rule, (iii.) by Simpson's rule (in each case dividing the area into six parts of equal breadths, and using tables to obtain the necessary ordinates), the area contained between this curve and the x axis.

2. A lake is found to be 1,600 yards across at its widest part; at intervals of 200 yards along a straight line, set out across it at this part, measurements are made perpendicularly to the line, and on either side of it, to the shore of the lake.

Estimate in two different ways the area of the surface of the lake when the measurements are (in yards):

Distance across	0	200	400	600	800	1,000	1,200	1,400	1,600
On one side ..	0	820	1,030	1,045	970	800	605	320	0
On other side..	0	350	305	275	240	215	250	230	0

3. A flat rectangular metal plate, OXAY, has a curve drawn across it from O to A. OX is 6 feet, OY is 7·2 feet, and the curve is drawn from the data:

$x=0$	1	2	3	4	5	6 feet
$y=0$	0·2	0·8	1·8	3·2	5·0	7·2 ,,

x being measured parallel to OX, and y to OY.

The part OAX is now cut away. Find the area of the remaining part OYA.

CHAPTER IV

APPROXIMATE DETERMINATION OF VOLUMES

§ 18. THE methods explained in the last chapter can be applied also to the approximate determination of the volumes of solids of revolution, and also to solids of certain other limited classes.

Consider the solid formed by the revolution of the curve AC (Fig. 27) about the straight line BD . Let the radius of the cross-section, at a distance x from B , be denoted by r .

If we draw what we called in § 13 the inner staircase for the curve AC , we replace the solid by a series of thin cylindrical plates, corresponding to the rectangle

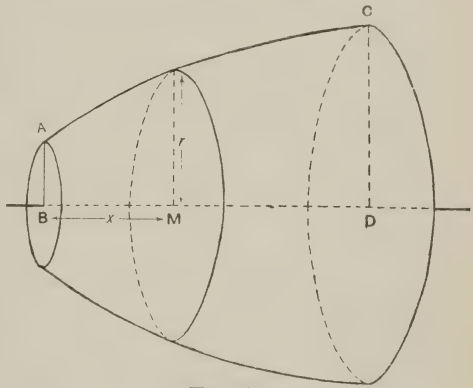


FIG. 27.

strips of Fig. 18. The sum of the volumes of these is an approximation to the volume of the solid. (See Fig. 28.)

We can represent this process thus:

Construct, on the base-line BD , a curve, the ordinates of which represent the areas of the cross-sections of

the solid. Thus, in Fig. 29, the ordinate PM is of length πr^2 units where BM is x , the connection between r and x in Fig. 27 being explained above.

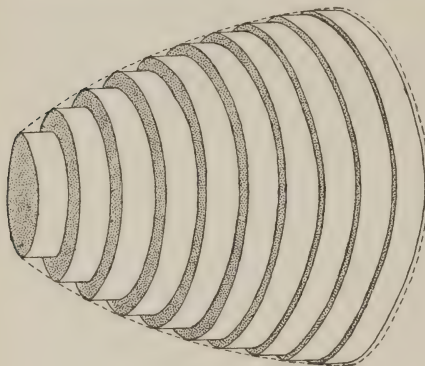


FIG. 28.

If a staircase be drawn for the curve in Fig. 29, the number of units of area in the rectangle-strip under each stair will be the same as the number of units of volume in the corresponding plate in Fig. 28.

Thus, the number and breadths of the stairs being the same, the sum of the areas of the rectangle-strips in Fig. 29 will be the same as the sum of the volumes of the plates in Fig. 28.

As the stairs are made narrower and their number increased, these sums approximate more and more nearly to the area under the curve and the volume

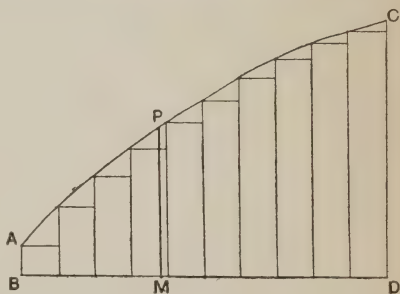


FIG. 29.

of the solid respectively. Thus we can say that the number of units of volume in the solid of revolution is

equal to the number of units of area under the curve formed by plotting the areas of cross-sections against the base-line—which is the axis of the solid.

Now this area can be found at once by any of the rules in Chapter III. In favourable cases, it is not even necessary to draw out the curve of cross-sections. If the radii of the cross-sections are given at equal intervals, we can square these and multiply by π . This gives us the ordinates of the curve sought, and we can continue as in Chapter III. We can simplify the calculation by deferring the multiplication by π to the end. This means that we find the mean ordinate of the r^2 curve, or, in other words, the “mean value of r^2 ” for the whole curve. We multiply this by π and by the length of the base-line; this gives the volume required.

Example.—The axis of a solid of revolution is 10 inches in length, and the radii of the cross-sections at 1, 2, 3, . . . 10 inches from the vertex are respectively 1.05, 1.75, 2.25, 2.55, 2.85, 3.00, 3.15, 3.25, 3.30, 3.30 inches. Estimate the volume of the solid.

We can consider the base-line divided into five portions; thus we can obtain the mean of the squares of the five mid-ordinates:

$$\begin{array}{r} 1.05^2 = 1.103 \\ 2.25^2 = 5.063 \\ 2.85^2 = 8.123 \\ 3.15^2 = 9.923 \\ 3.30^2 = 10.890 \end{array}$$

Sum 35.102

Mean value = 7.02 square inches.

Approximate volume = $\pi \times 7.02 \times 10$ cubic inches = 220 cubic inches.

Or we could have obtained what is probably a more accurate result, by using all eleven ordinates and applying Simpson's rule:

<i>Radii.</i>	<i>Squares.</i>	<i>Multipliers.</i>	<i>Products.</i>
0	0	1	0
1.05	1.103	4	4.412
1.75	3.063	2	6.126
2.25	5.063	4	20.252
2.55	6.503	2	13.006
2.85	8.123	4	32.492
3	9	2	18
3.15	9.923	4	39.692
3.25	10.56	2	21.12
3.3	10.89	4	43.56
3.3	10.89	1	10.89
		Total 30	Sum 209.55

Mean value of $r^2=6.98$ square inches.

Whence the approximate volume is 219 cubic inches.

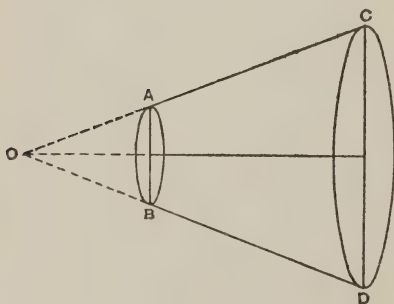


FIG. 30.

The application of this method to find the volume of a frustum of a cone is of especial interest. AB is a diameter of the top, and CD a parallel diameter of the bottom of the frustum (Fig. 30). O is the vertex of the cone, of which the frustum is a part. Then, if

r is the radius of the cross-section at a distance of x from O, $r=kx$, where k is the tangent of the semi-vertical angle of the cone. Therefore, $r^2=k^2 x^2$. So the curve obtained by plotting the areas of the cross-sections against the axis will be a parabola. Now for a parabola, the area-rule is known (Simpson's rule, in fact), so that the mean cross-section, and hence the volume, can be obtained at once exactly.

As an illustration, suppose the radii of top and bottom to be 1 foot and 1 foot 8 inches respectively, and the height of the frustum (*i.e.*, the length of its axis) to be 10 inches, the volume in cubic inches is given by the rule to be—

$$\begin{aligned} & \frac{10\pi}{6} (12^2 + 4 \times 16^2 + 20^2) \\ &= \frac{5\pi}{3} \times 1568 = 2613\pi = 8210 \text{ cubic inches} \\ &= 4.75 \text{ cubic feet.} \end{aligned}$$

EXAMPLES IV.

1. If A is the area of a section of a sphere of radius r , by a plane whose distance from the centre of the sphere is d , find an equation for A in terms of r and d . Sketch the shape of the graph, showing the relation between A and d , for values of d from $d=0$ to $d=r$, and show that Simpson's rule can be used to find the exact volumes of a spherical segment, and of a slice cut from the sphere by two parallel planes.

2. A sphere, centre O, is cut by two parallel planes on either side of the centre. If the radius of the

sphere is 13 inches, and the planes are distant 5 inches and 7 inches from O , find, using Simpson's rule, which gives such a volume accurately, the volume of the portion of the sphere cut off between the two planes. Find also a general formula for such a volume, taking the radius of the sphere to be R , and the distances of the two planes from the centre to be d_1 and d_2 . (Cf. Examples II., No. 27.)

3. A bowl is in the shape of a "solid of revolution"—*i.e.*, its horizontal sections are all circular. The following table gives the length of the radius (r inches) at a depth d inches below the top:

d ..	0	0.6	1.5	2.25	3.0	3.75	4.5	5.0	5.4	6
r ..	4	4.18	4.9	5.55	6.0	5.95	5.125	4.0	2.7	0

Plot r against d and find the volume of the bowl (*i.e.*, the volume of liquid that the bowl will hold when full. All measurements are internal).

4. The cross-sections of a tree (A square inches) at distance x feet from one end are as follows:

x ..	10	30	50	70	90	110	130	150
A ..	120	123	129	129	131	125	142	156

What is the volume of the tree in cubic feet, its length being 160 feet?

5. There is a natural reservoir with irregular sides.

When filled with water to a vertical height h feet above the lowest point, the following is the area A in thousands of square feet of the water surface:

$h \dots$	0	5	10	20	30	42	50	65	70
$A \dots$	0	220	322	435	505	560	586	617	624

Plot A against h on squared paper.

Find the mean value of A between $h=10$ and $h=65$.

6. The main body of a submarine is shaped by rotating an ellipse about its longer axis (which lies along the axis of x). The equation to the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a=60$, $b=8$. Calculate the value of y^2 for the following values of x : -55 , -45 , -35 , -25 , -15 , -5 , $+5$, $+15$, $+25$, $+35$, $+45$, $+55$.

Hence, by the mid-ordinate rule, find the area of the mean cross-section of the submarine, and hence its volume.

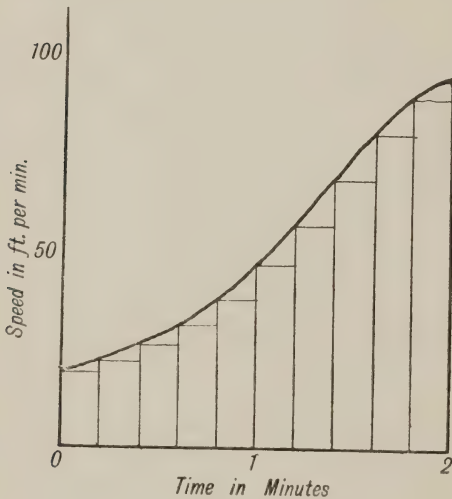
7. At low tide the water-surface of a certain dock has an area of 43,400 square yards, and the area changes to 45,320, 46,700, 48,200, 49,810, 50,500, 51,060, for each 3 feet the tide rises (total rise 18 feet). Find the volume of water that enters the dock between high and low tides.

CHAPTER V

APPLICATION TO PROBLEMS OF MECHANICS

§ 19. THE measurement of the area under a curve is of some importance in certain mechanical problems.

For instance, if we are given the speed-time curve which represents the motion of a body



along a line, the number of units in the area under the curve (due regard being paid to the units in terms of which the speed and time have been measured) represents the displacement in the given time.

For the explanation of

this, we must go back to the idea of our staircases.

Fig. 31 shows part of the speed-time graph of a body, the speed of which increases from 20 to 95 ft./min

in two minutes, in accordance with the law expressed by the graph.

We draw the "inner staircase" for breadths of one-fifth of a minute. Suppose that the change of speed, instead of being continuous, is in accordance with the staircase—*i.e.*, that the speed changes abruptly every fifth of a minute from 20 to 22·6 to 27·0, etc., ft./min., and remains uniform for the succeeding fifth of a minute. The area under each stair now represents the displacement during the time represented by the base-line under the stair. (For each stair corresponds to a uniform speed, and the height of the stair above the base-line—the ordinate—measures this speed. The product of the measure of this speed, and the time during which it persists, measures the displacement during the time.)

So the sum of these rectangles measures the total displacement corresponding to this imaginary, abruptly changing speed.

As we make the number of stairs greater, and the corresponding time-intervals shorter, the abrupt changes of speed merge, as it were, into the continuously changing speed. At the same time, the sum of the areas of the rectangles merges into the area under the curve. So we conclude that the area under the curve measures the displacement corresponding to the changing speed represented by the given curve.

This area can be found by whichever method of those explained in Chapter III. is most convenient.

Example 1.—The speed of a train, which starts from

rest, is given by the following table, the times being reckoned in minutes from the start, and the speed in miles per hour:

<i>t</i> minutes ..	0	2	4	6	8	10	12	14	16	18	20
<i>v</i> miles per hour	0	10	18	25	29	32	20	11	5	2	0

Estimate the total distance travelled in the twenty minutes.

Simpson's rule is applicable:

<i>Ordinates.</i>	<i>Multipliers.</i>	<i>Products.</i>
0	1	0
10	4	40
18	2	36
25	4	100
29	2	58
32	4	128
20	2	40
11	4	44
5	2	10
2	4	8
0	1	0
Total 30		Sum 464

Mean ordinate = 15.47 (units, miles per hour).

Area under curve = $15.47 \times 20 \div 60 = 5.1(6)$ miles.

The above mean ordinate, 15.47 miles per hour, is, in fact, the average speed of the train during the twenty

minutes of motion. Thus the distance described in the time is $15.47 \div 3 = 5.1(6)$ miles.

§ 20. In the same way, if we have an acceleration-time curve for a problem of motion, the area under it gives the increase in speed during the time represented by the base-line.

If, during the interval considered, the acceleration becomes negative—*i.e.*, a retardation—the curve will cross the time axis. The area below the axis must be reckoned negative, which implies a decrease of speed. Either the positive and negative areas should be evaluated separately, or the mean ordinate of the whole curve can be found by one of the rules, each ordinate in the table having its proper sign prefixed.

Since, by the second law of motion, $F = m \times a$, where F is a force measured in poundals (or dynes), m the mass in pounds (or grammes) of a body on which the force is impressed, and a the resulting acceleration in feet per second² (or cm. per second²), so, if we are given a curve in which the impressed force is plotted above a time-axis, the area under the curve represents m multiplied by the increase in the speed, or, in other words, the increase in momentum. Due regard, of course, must be paid to units.

Example 2.—A car weighing 6 tons starts from rest under the action of an impressed force given by the table on p. 66, and is resisted by a constant frictional force of 40 lbs. weight per ton. Find the speed of the car at the end of 20 seconds.

The constant frictional force is 240 lbs. weight. This

can either be subtracted from each of the values of the impressed force, as given in the table, and the resulting values of the accelerating force treated as the ordinates of the force-time curve, or we can find the mean ordinate of the curve given by the above values, and subtract 240 from that.

<i>t</i> seconds ..	0	2	4	6	8	10	12	14	16	18	20
<i>P</i> lbs. weight (impressed force) ..	780	750	708	620	495	420	365	324	300	280	270

Proceeding by this latter method, we obtain the mean ordinate by the mid-ordinate rule. This gives 479 lbs. weight for the mean ordinate.

Thus, the mean effective accelerating force is 239 lbs. weight. We multiply by 32 to reduce it to poundals, and divide by the mass (6×2240 lbs.).

This, therefore, multiplied by 20, gives the increase in speed in feet per second during 20 seconds. The initial speed is zero; therefore the speed at the end of 20 seconds is this value of the increase. It comes to 11.4 ft./sec.—*i.e.*, 7.8 miles per hour approximately.

§ 21. Another case, of much practical importance, arises when the force, acting on a body free to move, is plotted on a curve with the corresponding values of the displacement along the base-line.

In this case, adjacent sides of the rectangle-strip under a stair measure respectively a constant force, and the distance through which its point of application

is moved. Therefore, the area of the strip measures the work done during the displacement represented by the breadth of the stair.

Arguing in this way, we see that the total area under the force-displacement curve is a measure of the work done on the body during the total displacement.

Example 3.—A truck, starting from rest, and weighing 15 tons, is drawn along the level, against a constant resistance of 30 lbs. per ton. The draw-bar pull is found to vary with the distance travelled, in accordance with the following table:

Distance travelled in feet= s	0	10	20	30	40	50
Draw-bar pull in lbs. weight= F ..	900	890	868	822	763	679

Find the speed of the truck after it has gone 50 feet.

We will use the trapezium rule (this saves plotting the curve):

<i>Ordinates.</i>	<i>Multipliers.</i>	<i>Products.</i>
900	1	900
890	2	1,780
868	2	1,736
822	2	1,644
763	2	1,526
679	1	679
Total 10		Sum 8,265

Hence, the mean value of the draw-bar pull is 826.5 lbs. weight.

Subtracting 450 lbs. weight, the constant resisting force, we have:

Mean effective accelerating force = 376.5 lbs. weight.

This, multiplied by 50 feet, represents the area under the curve, and so the work done. This comes to 18,825 ft. lbs.

The amount the kinetic energy of the truck has increased is equal to this.

Hence, if v be the speed in feet per second acquired from rest at the end of the 50 feet, we have—

$$\frac{1}{2} \times \frac{15 \times 2240}{32} \times v^2 = 18825.$$

Whence v is about 6 ft./sec.

§ 22. It is well to be very clear as to the exact signification of the words “mean” and “average,” especially as these terms are often used in a confusing manner.

The arithmetic mean of a number of measures of the same kind is found by adding together these measures and dividing the sum by their number. The mid-ordinate rule gives us an example of an arithmetic mean.

The terms “mean” and “average” of a number of measures are often used as abbreviations for arithmetic mean.

The mean or average ordinate of the area under a

curve implies, as we have seen, an ordinate which, if placed at one end of, and at right angles to, the base-line, defines a rectangle, whose area is that of the area under the curve.

What is termed in mechanics the mean or average speed of a moving body during a given interval is the constant speed at which the body should move through the given interval in order to receive the same displacement as it receives under the actual conditions. This is seen to be the same as the measure (in units of speed) of the mean ordinate of the speed-time curve (§ 19).

But, referring to §§ 20 and 21, it will be noticed that in Examples 2 and 3, the same term—"mean effective accelerating force," or briefly "average force"—has been used in two distinct senses. In Example 2 it is measured by the mean ordinate of the force-time curve. In Example 3 it is measured by the mean ordinate of the force-displacement curve. In Example 2 it is that force which, multiplied by the total *time* elapsed, gives the change in momentum. In Example 3 it is that force which, multiplied by the total *displacement*, gives the work done. These two values of the average force are not the same in the same problem. Strictly speaking, they should be qualified as being "with respect to the time" and "with respect to the displacement" respectively.

To illustrate this, consider the case of a variable force given by $10t$ poundals (t is the number of seconds in the time) acting on a mass of 1 lb. starting from rest.

We have the following table:

Time in seconds, t ..	0	1	2	3	4	5	6
Force in poundals, F ..	0	10	20	30	40	50	60

The force-time graph is clearly straight. Therefore, the average force "with respect to the time" is 30 poundals, the mean acceleration is 30 ft./sec.², and the speed acquired after 6 seconds is 180 ft./sec.

If s is the displacement in feet after t seconds, it can be proved by Calculus that $s = \frac{10}{6} t^3$.

Hence the table:

Time in seconds, t	0	1	2	3	4	5	6
Force in poundals, F	0	10	20	30	40	50	60
Displacement in feet, s	0	1.67	13.33	45	106.7	208.3	360

F can now be plotted against s (this is left for the reader), and the mean ordinate of the curve found between $s=0$ and $s=360$.

From this we deduce that the average force with respect to the displacement "is 45 poundals. (Contrast this with the value, 30 poundals, above.)

From this result the work done is 45×360 foot-poundals. Therefore, if the speed acquired in 6 seconds is v ft./sec., we have:

$$\frac{1}{2} \times 1 \times v^2 = 45 \times 360,$$

$$v = 180 \text{ ft./sec. (as before),}$$

showing that the two values of the average force are in harmony.

§ 23. Fig. 32 is an "indicator diagram," such as is used to find the horse-power of steam and internal combustion engines.

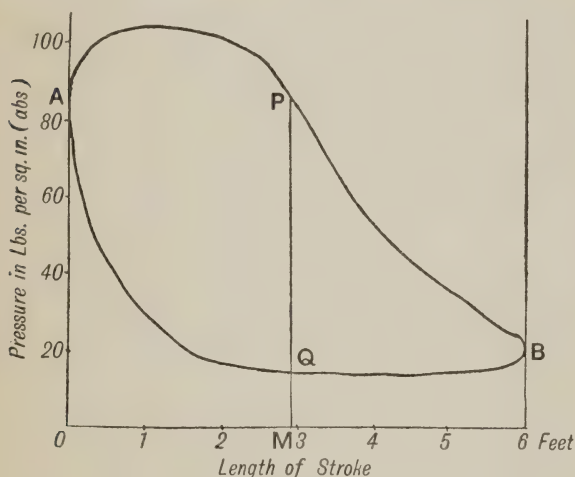


FIG. 32.

The base-line represents, on a reduced scale, the stroke of the piston. The piston moves forward from 0 to 6, then backwards from 6 to 0.

The ordinates represent the pressures of the fluid inside the cylinder, with the understanding that on the forward stroke the portion of the curve APB is

the pressure-curve, and on the backward stroke the portion BQA.

To get the force on the piston in any position, we should multiply the pressure for that position (lbs. per square inch) by the area of the piston face (square inches). Thus, the area under the curve APB, multiplied by the area of the piston face, gives the total work done on the piston by the fluid, on the outward stroke.

On the backward stroke the piston is doing work on the fluid, and the area under BQA, multiplied as before by the area of the piston face, measures this work.

Thus, the net work done on the piston during the stroke is measured by the area of the closed curve APBQA, multiplied by the area of the piston face.

To obtain the area of the closed curve, we may use an instrument called a planimeter, or we may proceed, as shown in Fig. 23, by the mid-ordinate rule. We divide the base-line into 10 (or a convenient number of) equal parts, measure the middle-breadths corresponding to these, and find their arithmetic mean. This is called the mean effective pressure, corresponding to the diagram, and, multiplied by the length of the stroke and by the area of the piston face, gives the work done on the piston per stroke.

§ 24. In all such mechanical problems as those illustrated above, it is necessary to be on one's guard against evaluating the area between the curve and the wrong axis.

For instance, if in Fig. 31 the speeds had been plotted along a horizontal, and times along a vertical axis, the area "under the curve" would not give the displacement sought, though, since the mean height of the curve would represent a time, and the base-line a speed, the area would presumably measure a displacement of sorts. The only safe way to determine which of the two possible areas gives the result desired, in any such problem, is to consider the physical meaning of the area under a staircase drawn in connection with the curve. In the problem above mentioned, a rectangle of a staircase drawn on the wrong side of the curve—*i.e.*, down to the speed axis—would have for one of its sides a length representing the "date" or epoch, and not a duration of time, and for its other side a small *increase* of speed, and not the total speed with which the body is moving. The product of these can have no physical meaning.

EXAMPLES V.

1. A body is moving along a straight line in such a way that its speed (v ft./sec.), at the end of t seconds, is given by the formula $v=t^2(20-t)$. Plot the graph showing v as a function of t between the values $t=10$ and $t=16$.

Estimate the distance in feet the body travels in this interval of time, and find its average speed during the interval.

2. A certain body is moving along a straight line in such a way that its speed v (inch/sec.) at the

end of t seconds is given by the equation $v=a \sin kt$. If $a=120$, $k=9$, and the angle is measured in degrees, show that the body will first come to rest after twenty seconds. How far will the body have moved up to this point?

3. Air, compressed in a cylinder, expands from 2 cubic feet to 10 cubic feet, the pressure p , measured in atmospheres, being given by the following table:

v	2	4	6	8	10
p	18.8	7.07	4.00	2.66	1.95

Find the work done in foot-pounds by the expansion of the air (1 atmosphere=14.7 lbs. per square inch).

4. A gas expands in such a way that the pressure (p lbs. per square inch), and the volume (v cubic feet) are connected by the law $pv=100$. Calculate the work done by the gas in expanding from 3 cubic feet to 9 cubic feet.

5. The pull F lbs.-weight on a tram-car was registered when the car was at the following distances, x feet, from a certain point:

x	0	12	20	34	50	62
F	610	480	460	450	470	490

What is the total work done on the car in moving it through the distance of 62 feet?

6. The speed of a tube-train for the first ninety seconds of its motion is given by the following table:

Time in seconds from start ..	0	5	10	20	30	40	50	60	70	80	90
Speed in ft. per sec. ..	0	4	8.8	18.5	27	31.5	33	29.5	23.5	20	18.5

Find the distance travelled in the given period of ninety seconds.

7. The relation between the pull on a truck and the distance travelled from the starting-point is given by the following table:

Distance in feet ..	0	100	200	300	405	485	555	620	700	760	860	940	1,000
Pull in lbs. weight..	500	540	540	520	480	440	400	360	300	260	200	160	134

Find the total work done on the truck as it is moved over the 1,000 feet.

8. The following results were obtained during a tensile test on a steel bar:

Load in tons..	0	1	2	3	4	5	6	7	8	9	10
Elongation in thousandths of an inch ..	0	1.1	2.2	3.2	4.3	5.5	6.7	7.8	12.0	19.4	28.0

Plot these values against one another, and draw a smooth curve through the points so obtained; hence, find the work done in inch-tons during the test.

9. The following table gives the resistance of the air in lbs.-weight, at intervals of 150 feet, to the motion of a shell when flying horizontally:

648.5	640.5	632	610	588	568	549.5	536	518
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Find in foot-pounds the work done for the whole distance of 1,200 feet.

10. The net pull (P) on a tram-car (*i.e.*, taking into account the retarding frictional forces) at intervals of ten seconds, is given by the following table:

t seconds	..	0	10	20	30	40	50	60	70	80
P lbs. weight	..	60	64	63.8	61.5	54.2	44	36	29.8	25.2

The mass of the car is 5,000 lbs., and it starts from rest. Calculate the speed (ft./sec.) with which it is moving at the end of eighty seconds.

11. The following table gives the pressure p (lbs. per square inch), and the volume v (cubic feet) of saturated steam:

v	..	7	7.6	8.4	9.2	10.3	11.7	13.5	16.1	18.1	20
p	..	60	55	50	45	40	35	30	25	22	20

Find the work done (in foot-pounds) by the steam, in expanding from 8 cubic feet to 20 cubic feet.

12. A car starts from rest, and the following table gives the distances passed over, and the corresponding tractive forces exerted by the motor:

Tractive force in lbs. weight	850	800	750	700	650	600	550
Distance in feet ..	0	34	48	57	65	75	100

There is a constant resistance due to friction and other causes of 200 lbs.-weight. Calculate the kinetic energy of the car at the last observation.

13. (i.) If a body moves with uniform acceleration a ft./sec. per second, show that if the reciprocal of a be plotted up the vertical axis, and the speed v (ft./sec.) along the horizontal axis, then the area of the rectangle contained between the straight line forming the graph, the horizontal axis, and ordinates corresponding to two values v_1 and v_2 of the speed, measures the time in seconds for the speed of the body to rise from v_1 to v_2 .

(ii.) If the acceleration is not constant, the graph of $1/a$ may be a curved line. By a "staircase" argument, similar to that employed in the text of this chapter, show that the area under this curve, down to the speed axis, represents the time taken for a given increase of speed.

(iii.) The tractive force exerted by the engine of a racing car may be taken as constant for all speeds,

and equal to 700 lbs.-weight. The resistance, due to friction, air, and other causes, may be taken as $50 + 6 \left(\frac{V}{10} \right)^2$ lbs.-weight, where V is the speed in miles per hour. Work out the net accelerating force on the car for speeds of 0, 10, 20, . . . 100 miles per hour. Show that the maximum speed attainable is about 103 miles per hour. Taking the mass of the car to be 1 ton, calculate the values of $1/a$ (a being in ft./sec. per second), corresponding to the above values of V . Find the mean value of $1/a$ between $V=0$ and $V=90$ miles per hour. Hence, find the time taken for the car to acquire a speed of 90 miles per hour from rest. Go on to compute the extra time taken for the speed to rise from 90 to 100 miles per hour. Notice that the car is approaching its (theoretical) maximum speed asymptotically.

Note.—This process of plotting the reciprocal of the acceleration as a function of the speed, and of finding its mean value over a given speed-range, can be applied to many problems of practical importance. The student of Mechanics will find further information on this point in Ashford's *Elementary Dynamics*, 2nd Edition, p. 233, *et. seq.*

APPENDIX I

THE SURFACE AREA OF A ZONE OF A SPHERE

IN Examples I., No. 21, p. 15, it is shown that the curved surface of a frustum of a cone, of which the radii of top and bottom are r_1 and r_2 and the slant l , is given by the formula $\pi(r_1 + r_2) \times l$.

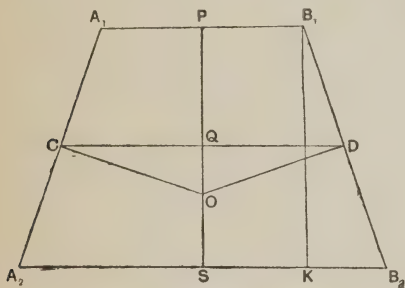


FIG. 33.

Fig. 33 represents a section through the axis of a frustum of a cone. CD bisects A_1A_2 and B_1B_2 , and it is obvious that $QD = \frac{1}{2}(r_1 + r_2)$. Therefore the area of the curved surface of the frustum

is $2\pi \times QD \times l$. (This follows directly from Pappus' second theorem. See Examples II., No. 34, p. 28.)

If DO, CO, perpendiculars to B_1B_2 and A_1A_2 , meet on PQS at O, it follows that—

$$\frac{QD}{OD} = \cos QDO = \sin B_1B_2K = \frac{B_1K}{l}.$$

$$\therefore QD \times l = OD \times B_1K.$$

$\therefore$ Area of curved surface $= 2\pi OD \times h$ (where h is the height of the frustum).

In Fig. 34, imagine the whole surface of the sphere indicated to be surrounded by a series of small frustums of cones, each touching it along a small circle, the plane of this small circle being half-way between the upper and lower faces of the frustum. Thus, the frustum represented in section in the figure by LHMN touches the sphere along the small circle shown by F_1G_1 .

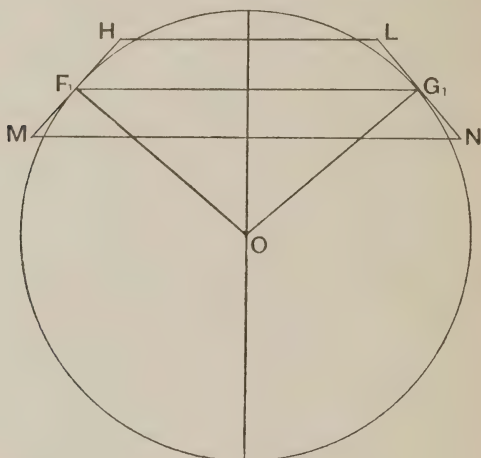


FIG. 34.

The area of the curved surface of the frustum is, by the above, $2\pi OG_1 \times h$.

O is now the centre of the sphere, and OG_1 is therefore the radius r ; h is the "height" of the frustum.

If we have any finite zone of the sphere, we can surround the portion of the surface of the sphere by a large number of such frustums, all touching the sphere along adjacent, parallel, small circles. By a

suitable choice we can, if necessary, arrange these frustums so that the lower face of any one of them exactly coincides with the upper one of that immediately below it, but our argument is equally sound whether we do this or not. (The process is analogous to that by which we deduced the area of a sector of a circle, in § 3, by surrounding the arc by a large number of tangent lines and using the "triangle rule" for each.)

By addition of the results, we find that the sum of the areas of the curved surfaces of the frustums is $2\pi r(h_1 + h_2 + h_3 \dots)$, where $h_1, h_2, h_3 \dots$ are the heights of the frustums.

The sum of $h_1, h_2, h_3, \dots$ is H , the total height of the zone.

As we increase the number of frustums which circumscribe the zone, the difference, if any, between the sum of their areas and the area of the curved surface of the zone of the sphere will get smaller and smaller.

So, we can say that the area of the curved surface of the zone of the sphere is $2\pi r \times H$.

But this is the same as the curved surface, contained between the same parallel planes, of the cylinder which circumscribes the sphere, touching it along the circumference of the great circle parallel to the bounding planes of the zone.

(Note.—The above result shows, in fact, that the sum of the surface areas of the frustums surrounding the zone, as in Fig. 34, is *exactly* equal to the surface area of the zone, however many, or however few, we take.)

APPENDIX II

TWO ADDITIONAL RULES FOR APPROXIMATING TO THE AREA UNDER A CURVE

THERE have been devised other rules for obtaining the mean ordinate of a curve, and hence the area under a curve, in addition to those given in Chapter III. Two of the most interesting of these follow:

1. *Weddle's Rule*.—In this rule the base-line is divided into six equal parts. We have thus 7 ordinates in all. Treat these as for the application of Simpson's rule, but with multipliers 1, 5, 1, 6, 1, 5, 1, instead of Simpson's multipliers.

This rule is especially useful, and accurate when we are dealing with curves drawn from equations involving trigonometrical ratios, when the angle-range is 180° or π radians.

Suppose we have the graph of $y = \sin x$ drawn from 0 to 180° . Length of base-line, 5 inches; unit for y , 1 inch.

To get the mean ordinate we have the following table:

<i>Angles.</i>	<i>Ordinates.</i>	<i>Multipliers.</i>	<i>Products.</i>
0°	0	1	0
30°	0.5	5	2.5
60°	0.866	1	0.866
90°	1	6	6
120°	0.866	1	0.866
150°	0.5	5	2.5
180°	0	1	0
		Total 20	Sum 12.732

Whence mean ordinate = 0.6366.

Area required = 3.183 square inches.

(Cf. Examples III.f, No. 1.)

In certain physical problems it is necessary to evaluate the area under the curve, $y = \frac{\sin x}{x}$, x being in radians, the range for x being 0 to π .

Here, again, it is convenient to use Weddle's rule. (The great advantage is that we need not use trigonometrical tables, since the sines of 30° and 60° alone are involved.)

The ordinates are: $1, \frac{0.5 \times 6}{\pi}, \frac{0.866 \times 3}{\pi}, \frac{1 \times 2}{\pi}, \frac{0.866 \times 1.5}{\pi}, \frac{0.5 \times 1.2}{\pi}$, and 0.

Before applying the rule, we will multiply throughout by π . Thus we have:

<i>Ordinates, each Multiplied by π.</i>	<i>Multipliers.</i>	<i>Products.</i>
3.142	1	3.142
3.0	5	15
2.598	1	2.598
2	6	12
1.299	1	1.299
0.6	5	3.0
0	1	0
	Total 20	Sum 37.039

Hence mean ordinate $= 1.852 \div \pi$.

Area required $= 1.852$ square units.

2. *Duften's Rule.*—The rule was first suggested in this form by Mr. A. F. Duften, in a letter to *Nature* of May 20, 1920. It may be expressed as follows:

Divide the base-line into a number of parts, which

is a multiple of 5. Draw specially indicated ordinates at the beginning and end of each set of 5 (*i.e.*, the first, sixth, eleventh . . . ordinates). These are shown in the figure by dotted lines. Take the arithmetic mean, not of these, but of the ordinates next to and on each side of these bounding ordinates.

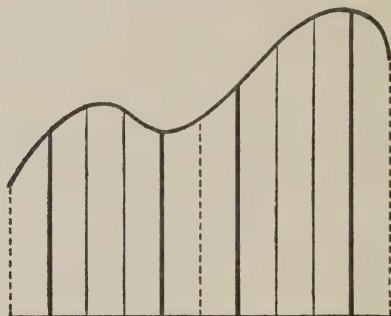
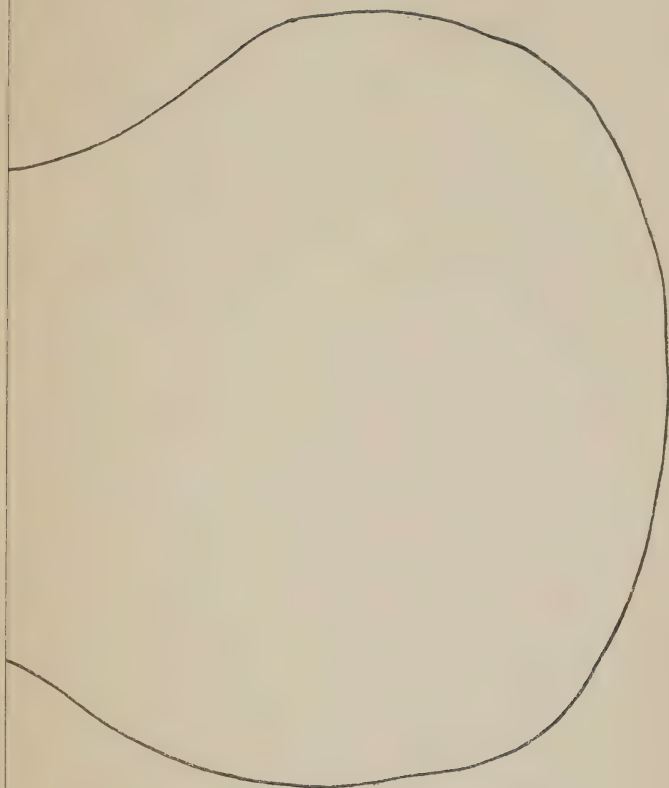


FIG. 35.

Thus, for ten ordinates, we take the second, fifth, seventh, and tenth. These are shown in the figure in thick lines. The arithmetic mean of these is an approximation to the mean ordinate of the curve.

If we apply this to the quadrant of the circle considered in Chapter III., the work is as follows: The ordinates selected (see § 15, p. 46) are 4.359, 8, 9.165, 9.95. The arithmetic mean of these is 7.87, and the area, therefore, 78.7 square cm.

This rule is easy to apply, and gives uniformly good results.

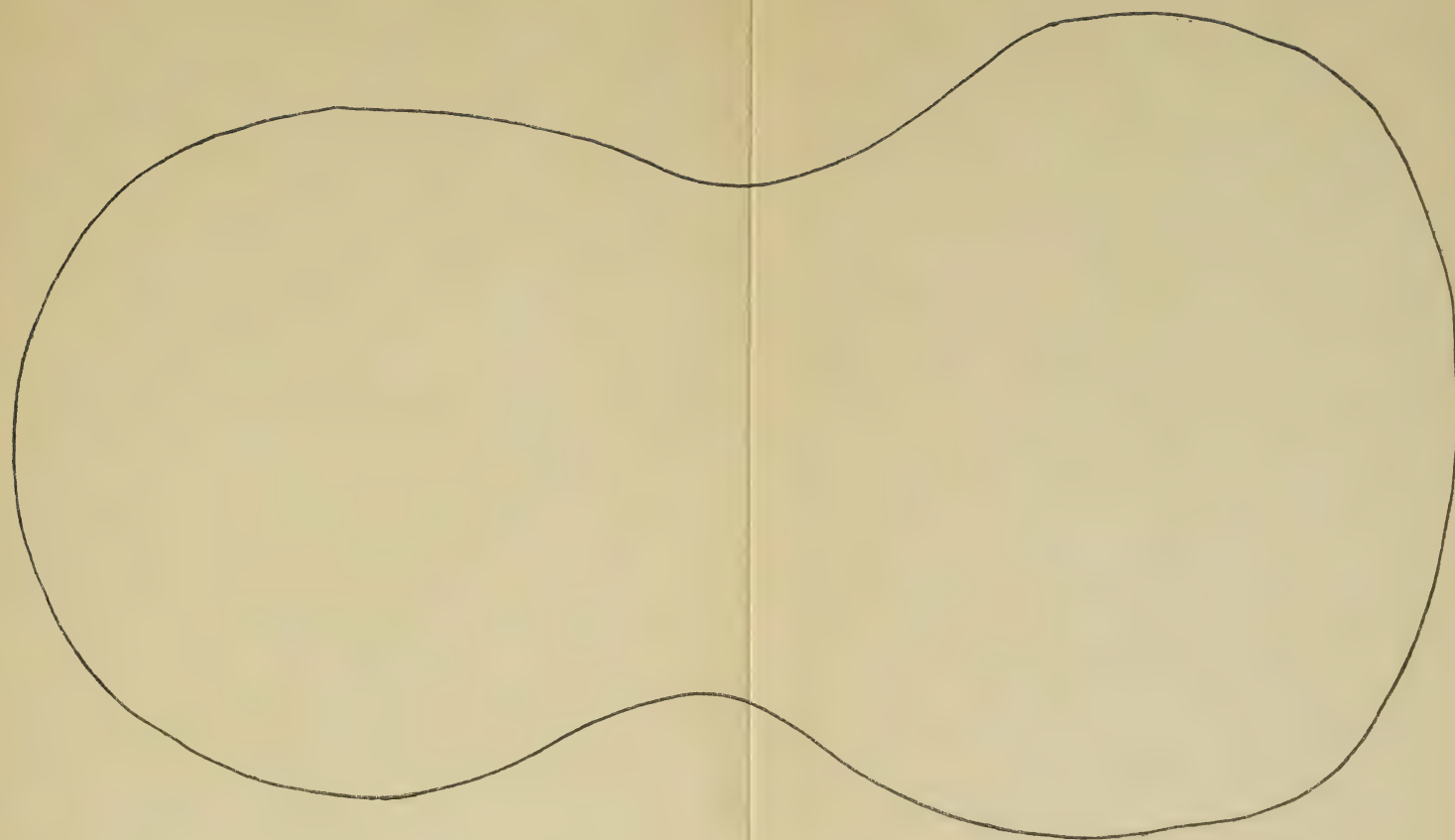


CURVE FOR EXAMPLES :

IIIc, No. 7, page 44.

IIId, No. 7, page 48.

To face page 84.



CURVE FOR EXAMPLES:
IIIc, No. 7, page 44.
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ANSWERS TO EXAMPLES

EXAMPLES I

Page 13—

1. $0.433a^2$ sq. in.
2. $0.433a\sqrt{a^2 - x^2}$.
3. 0.50%
4. $ab - 0.8584r^2$ sq. cm.
5. $15.6(2)$ sq. in.; $6.00(7)$ in.
7. $2n - 1$; n^2 .

Page 14—

8. $2.19(2)$ in.
9. $2.828r^2$; $3.31(4)r^2$.
10. 306 sq. in.
11. $a^2 + a\sqrt{4b^2 + a^2}$.
12. $3a + b$; $a + 3b$
13. $3.50(5)$ in.; $9.11(3)$ sq. in.

Page 15—

15. $2.16(8)$ sq. in.
16. $\frac{rl}{2} + \frac{l^2}{4\pi}$; $2 \sin^{-1} \frac{l}{2\pi r}$
17. $58.7(9)$ sq. in.
18. $127.3(3)$ sq. ft.; $271.6(6)$ sq. ft.
19. $\sin \frac{\theta}{2} = \frac{a}{2\pi}$; $\theta = \frac{\pi}{3}$ (i.e. 60°)
20. $\frac{l(r_1^2 - r_2^2)}{2r_1}$

Page 16—

22. 31 in.; 43.3%
23. $45.9(3)$ sq. in.
25. (i.) $169.6(6)$ sq. in.; (ii.) $275.6(6)$ sq. in.
26. 85.0 sq. in.
27. $7.06(8)$ sq. ft., yellow; $5.17(4)$ sq. ft., blue; $1.89(4)$ sq. ft., red.

EXAMPLES II.

Page 23—

1. $\frac{a^2b}{4\pi}$ cu. in.; $\frac{ab^2}{4\pi}$ cu. in.
2. (i.) $\frac{a^3}{6}$; (ii.) $0.866a^2$; (iii.) $0.5774a$.
3. $\frac{l^3\theta^2}{12\pi} \sqrt{1 - \frac{\theta^2}{4\pi^2}}$
4. No.
5. $r = \sqrt{\frac{d^3}{6l}}$.
6. $21.3(4)$ in.; $10.8(7)$ cu. ft.

Page 24—

7. $89.7(2)$ lb.
8. $2.64(6)$ in.
9. Volume of cylinder: volume of cone : : 0.7025 : 1
: : 1 : 1.423 .
10. $0.66(6)$ in.
11. $25.1(3)$ cu. ft.
12. 452.4 cu. in.; 137.5 cu. in.

Page 25—

13. $5.781s^2$
16. $241\frac{2}{3}$ cu. ft.
17. $29.2(3)$ cu. ft.

Page 26—

18. 390 cu. ft. 19. 7.95(2) cu. in. 20. 0.717(5) pints.

21. $\frac{S}{V} = 2\left(\frac{1}{l} + \frac{1}{r}\right)$; 263.2 sq. ft.

Page 27—

24. $\frac{\pi}{6} (2r^2H + r^2h + h^3)$. 25. 829.6 cu. ft.

26. 7.54 cu. in. 28. 4.35 in.

Page 28—

29. $4\pi^2 = 39.48$ cu. in.; 1 ton. 31. $\frac{4r}{3\pi}$; $\frac{2r}{\pi}$.33. $3\frac{1}{9}$ in. from AB; $2\frac{2}{3}$ in. from BC.

EXAMPLES IIIa.

Page 32—

The correct answers, by theory, are:—

1. 9.16 sq. in. 2. 5.63 sq. in.

EXAMPLES IIIb.

Page 35—

1. BE = 0.380(9)a; DE = 0.619(1)a. 2. (i.) $\frac{1}{3}$; (ii.) $\frac{1}{3}$.3. $7\frac{1}{3}$.

4. 4.225 in.

EXAMPLES IIIc.

Page 43—

1. 17 sq. units; “correct” result, 17.07. 2. 9.14 sq. in.

3. 5.63 sq. in.

Page 44—

4. 26.9 sq. ft. 5. About 6,400 sq. ft.

6. About 7 sq. in. (“correct” result, 6.73 sq. in.).

7. About 21.4 sq. in.

Page 45—

8. About 2.7 sq. in. and 8.0 sq. in. [“Correct” results, 2.66(7) and 7.87(5)]

EXAMPLES IIId.

Page 47—

1. 19,440 sq. ft. 2. 48.3 sq. ft.; 4,780,000 gallons per hour.

Page 48—

3. 230,000 cu. ft. per minute 4. 17.2 sq. units.

5. 9.21 sq. units. 6. 5.57 sq. in. 7. 21.4 sq. in.

Page 52—

EXAMPLES IIIe.

1. 1,320 sq. ft. 2. 2,960 sq. ft.; 7 tons per inch.

Answers to III*d.* worked by Simpson's rule:—

1. 19,640 sq. ft. 2. 48 sq. ft.; 4,750,000 gallons per hour
 3. 230,000 cu. ft. per minute. 4. 17.07 sq. units.
 5. 9.16 sq. units. 6. 5.63 sq. in. 7. 21.4 sq. in.

Page 54—

EXAMPLES III*f.*

1. (i.) 19.3 sq. in.; (ii.) 18.7 sq. in.; (iii.) 19.1 sq. in.
 (Correct answer, by theory, 19.099).
 2. About 1,500,000 sq. yd. (310 to 320 acres).
 3. 28.8 sq. ft. (The curve drawn is a parabola $y=0.2x^2$,
 which, therefore, divides the area of the plate in the
 ratio of 2 : 1.)

Page 59—

EXAMPLES IV.

1. $A = \pi(r^2 - d^2)$.

Page 60—

2. $1,872\pi = 5,880$ cu. in.; $\pi[R^2(d_1 + d_2) - \frac{1}{3}(d_1^3 + d_2^3)]$.
 3. Between 440 and 450 cu. in. 4. 146 cu. ft.

Page 61—

5. Between 520 and 530.
 6. 134 sq. ft.; 16,100 cu. ft. (The theoretical volume is
 $\frac{4}{3}\pi ab^2$, which comes to 16,080.) 7. 287,900 cu. yd.

Page 73—

EXAMPLES V.

1. 6,756 ft.; 1,126 ft. per sec. (These are the "correct" results.)

Page 74—

2. 127 ft. 3. 95,500 ft.-lbs. 4. 15,800 ft.-lbs.
 5. About 29,500 ft.-lbs.

Page 75—

6. About 2,010 ft. 7. Between 390,000 and 400,000 ft.-lbs.
 8. 0.204 in.-tons.

Page 76—

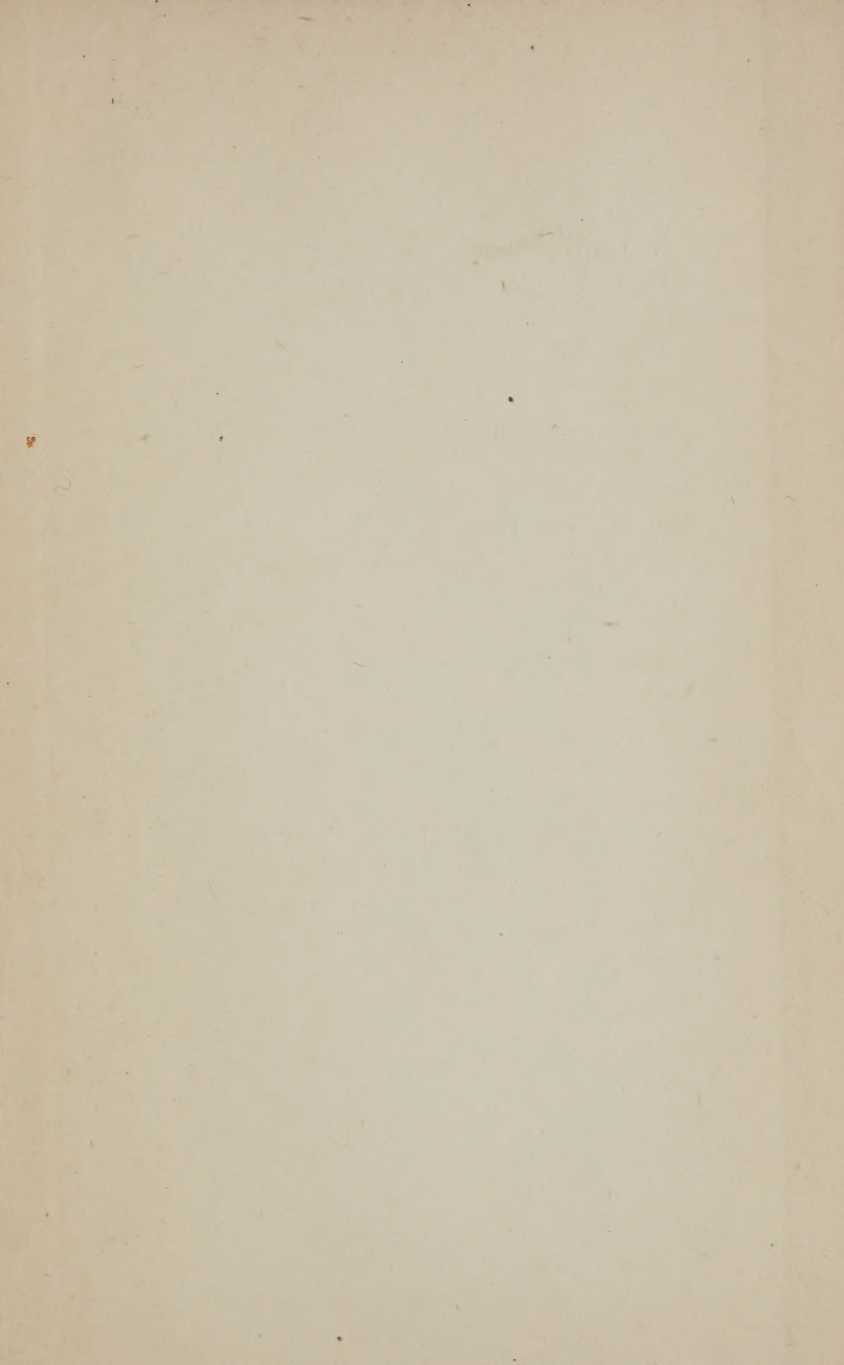
9. 706,000 ft.-lbs. 10. About 25.5 ft. per sec.
 11. Between 53,000 and 54,000 ft.-lbs.

Page 77—

12. About 51,500 ft.-lbs.
 13. Mean value of $\frac{1}{a} = 0.164$; time to acquire 90 miles per
 hour, 21.7 sec.; further time taken to acquire 100 miles
 per hour, 10.6 sec.

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